The Archimedean Spiral

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1 Introduction

1.1 Overview

Found in a variety of natural and human-made structures, the Archimedean Spiral is a geometric shape known for its aesthetic appeal and practical functionality.

1.2 History of Definition

In his comprehensive treatises *On Spirals*, Archimedes defined the Archimedean spiral as follows:

> If a straight line drawn in a plane revolves at a uniform rate about one extremity which remains fixed and returns to the position from which it started, and if, at the same time as the line revolves, a point moves at a uniform rate along the straight line beginning from the extremity which remains fixed, the point will describe a spiral in the plane.

As this definition was extremely wordy and lengthy, it was eventually altered.

In the 20th Century, mathematicians Alexander Ostermann and Gerhard Wanner modernized and simplified this definition in their book *Geometry by Its History*:

> The locus of points corresponding to the locations over time of a point moving away from a fixed point with a constant speed along a line that rotates with constant angular velocity.

2 Geometric Interpretation

2.1 Equation Derivation

Our goal is to derive the parametric equations defining an Archimedean spiral.

Theorem 1 *The parametric equations for the Archimedean spiral are*

$$x = (a + bt)\cos t, \qquad y = (a + bt)\sin t.$$

Let r denote the radial distance from a fixed point (the origin). By the definition of the Archimedean spiral, as the point moves further away from the origin, its angle θ increases proportionally to the radius:

$$r \propto \theta \quad \implies \quad r = b\theta.$$

To account for an initial offset from the origin, we introduce a constant a:

$$r = a + b\theta.$$

Converting from polar to rectangular coordinates:

$$x = r\cos\theta = (a + b\theta)\cos\theta, \qquad y = r\sin\theta = (a + b\theta)\sin\theta.$$

Substituting the parameter t for θ gives the final parametric equations:

$$x = (a + bt)\cos t, \qquad y = (a + bt)\sin t.$$

2.2 Image

An example of an Archimedean Spiral is shown below:

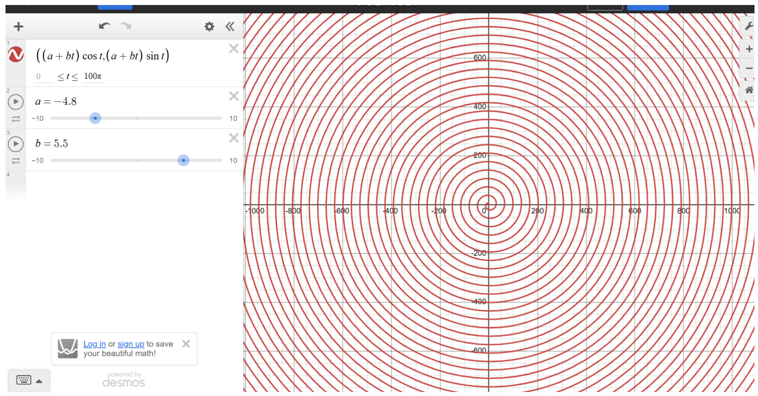

Figure 1: An Archimedean Spiral created with $a = -4.8$ and $b = 5.5$
(Try staring at the center for 30 seconds, then staring at the outside edge. An optical illusion should appear.).

2.3 Practice Problems and Solutions

Parametric Form of the Archimedean Spiral

The Archimedean spiral with the origin as the fixed point is defined by

$$x(\theta) = b\theta\cos\theta, \qquad y(\theta) = b\theta\sin\theta, \qquad \theta \in [0, 2\pi].$$

Problem 1: Arc Length in Parametric Form

Problem. Find the arc length of the Archimedean spiral from $\theta = 0$ to $\theta = 2\pi$.

Solution. The arc length is

$$L = \int_0^{2\pi} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta.$$

We compute derivatives:

$$\frac{dx}{d\theta} = b\cos\theta - b\theta\sin\theta, \quad \frac{dy}{d\theta} = b\sin\theta + b\theta\cos\theta.$$

Thus,

$$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = b^2(\theta^2 + 1),$$

so

$$L = \int_0^{2\pi} b\sqrt{\theta^2 + 1}\, d\theta.$$

Using

$$\int \sqrt{x^2 + 1}\, dx = \frac{x}{2}\sqrt{x^2 + 1} + \frac{1}{2}\ln(x + \sqrt{x^2 + 1}) + C,$$

we get

$$\boxed{L = b\left[\pi\sqrt{(2\pi)^2 + 1} + \frac{1}{2}\ln\left(2\pi + \sqrt{(2\pi)^2 + 1}\right)\right]}.$$

Problem 2: Tangent Slope at a Point

Problem. Find the slope of the tangent line at $\theta = \pi$.

Solution. The slope is

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\sin\theta + \theta\cos\theta}{\cos\theta - \theta\sin\theta}.$$

At $\theta = \pi$:

$$\frac{dy}{dx} = \frac{0 + \pi(-1)}{-1 - 0} = \pi.$$

$$\boxed{\text{Slope of the tangent at } \theta = \pi \text{ is } \pi.}$$

Problem 3: Distance Between Two Points One Turn Apart

Problem. Find the distance between points corresponding to θ_0 and $\theta_0 + 2\pi$.

Solution. The points are

$$P_1 = (b\theta_0\cos\theta_0,\, b\theta_0\sin\theta_0), \quad P_2 = (b(\theta_0 + 2\pi)\cos\theta_0,\, b(\theta_0 + 2\pi)\sin\theta_0).$$

Hence, the distance is

$$d = \sqrt{(2\pi b)^2(\cos^2\theta_0 + \sin^2\theta_0)} = 2\pi b.$$

$$\boxed{d = 2\pi b}.$$

Problem 4: Area Enclosed by the Spiral Using Parametric Form

Problem. Compute the area enclosed by the spiral from $\theta = 0$ to $\theta = 2\pi$ using

$$A = \frac{1}{2}\int_0^{2\pi} (x\,dy - y\,dx).$$

Solution. We have

$$dx = (b\cos\theta - b\theta\cos\theta)\,d\theta, \quad dy = (b\sin\theta + b\theta\cos\theta)\,d\theta.$$

Then

$$\begin{aligned} x\,dy - y\,dx &= (b\theta\cos\theta)(b\sin\theta + b\theta\cos\theta) - (b\theta\sin\theta)(b\cos\theta - b\theta\sin\theta) \\ &= b^2\theta^2(\cos^2\theta + \sin^2\theta) = b^2\theta^2. \end{aligned}$$

Thus,

$$A = \frac{1}{2}\int_0^{2\pi} b^2\theta^2\,d\theta = \frac{b^2}{2}\cdot\frac{(2\pi)^3}{3} = \frac{4b^2\pi^3}{3}.$$

$$\boxed{A = \frac{4b^2\pi^3}{3}}$$

3 Applications

3.1 Practical Properties

1. **Equal Spacing Between Turns.**
   The distance between successive arms is constant:
   $$\Delta r = b(2\pi).$$
   This uniform spacing ensures smooth motion.

2. **Linear Growth of Radius.**
   Since r increases linearly with θ, the spiral expands outward at a constant rate.

3. **Simple Mathematical Construction.**
   The spiral's linear form $r = a + b\theta$ allows for easy geometric and computational generation

4. **Uniform Area Coverage.**
   Each turn covers equal radial increments, ensuring even spatial coverage.

5. **Predictable Scaling.**
   Scaling the spiral by a constant factor preserves its overall shape,

3.2 Real-World Applications

1. **Archimedean Screw**
   The Archimedean Screw is a physical machine that utilizes an Archimedean spiral path to lift water. As the screw rotates, water from the bottom is lifted through the path to the top.

2. **Spiral Antenna**
   Spiral Antennas are a type of radio antenna known for their wideband operation, high spectral efficiency, and consistent performance over large frequency ranges. Due to the constant spacing between terms, the antenna operates at a variety of frequency ranges, with higher ranges corresponding to the inner spiral arm.

3. **Galaxy Classification**
   Although most spiral galaxies are logarithmic, Archimedean spirals have often been utilized to approximate these galaxies to a small degree of error.

4. **Coils**
   The Spiral is used to construct approximately equally-space coils on watches. This function applies in many other spring-based objects, highlighting the Spiral's wide-spread applicability.

5. **Coverage Path Planning** Coverage Path Planning is the task of determining an efficient path for a robot to completely traverse and cover an area of interest. The Archimedean Spiral creates a continuous, space-filling trajectory that covers a defined area.

6. **Diagnosis of Disorders** Doctors ask Patients to draw an Archimedean Spiral. Using these drawings, they are oftentimes able to observe and diagnose both the progression and presence of numerous Neurodegenerative movement disorders, including Parkinsons, Alzheimer's, and Essential Tremors (ET)

7. **Art and Architecture** The Creation of Spiral staircases and Spiral art, most notably found in the Vatican Museum, coupled with the usage of Archimedean Spirals to simulate whirlpools and tornadoes in digital Art, convey the curve's artistic beauty.

4 Related Math Problems

4.1 Coverage Path Planning

Theorem 2 *For a robot moving at constant linear speed along an Archimedean spiral $r(\theta) = b\theta$, the centripetal acceleration diverges as the spiral approaches the origin:*

$$\lim_{r\to 0} a = \infty.$$

Let the robot move at constant speed v along the spiral. The centripetal acceleration in polar coordinates is

$$a_c = \frac{v^2}{r}.$$

Since $r(\theta) = b\theta$, as the robot approaches the origin, $r \to 0$. Then

$$\lim_{r\to 0} a_c = \lim_{r\to 0} \frac{v^2}{r} = \infty.$$

Assuming negligible tangential acceleration a_t (constant speed), the total acceleration is

$$a = \sqrt{a_c^2 + a_t^2} = a_c \to \infty \quad \text{as } r \to 0.$$

Hence, the Archimedean spiral path causes infinite acceleration near the origin, which is physically impossible for a real robot.

Solution 1: Offsetting the Radius

If we start the spiral with a radius $r_0 > 0$, then the centripetal acceleration becomes

$$a_c = \frac{v^2}{r_0 + b\theta}.$$

Hence,

$$\lim_{\theta\to 0} a_c = \frac{v^2}{r_0} < \infty.$$

Solution 2: Velocity Profiling Along the Spiral

To maintain feasible acceleration, we can vary the angular speed $\dot{\theta}$ along the spiral.

Given the parametric coordinates of the Archimedean spiral:

$$x(\theta) = b\theta\cos\theta, \quad y(\theta) = b\theta\sin\theta,$$

$$x'(\theta) = -b\theta\sin\theta + b\cos\theta, \quad y'(\theta) = b\theta\cos\theta - b\sin\theta.$$

The velocity magnitude is

$$v = \sqrt{x'(\theta)^2 + y'(\theta)^2} < v_{\max},$$

where $v_{\max}$ is a design parameter.

This leads to the angular speed constraint:

$$b\dot{\theta}\sqrt{1+\theta^2} \leq v_{\max} \quad \Longrightarrow \quad \dot{\theta}(\theta) = \frac{v_{\max}}{b\sqrt{1+\theta^2}}.$$

The corresponding time function along the spiral is

$$t(\theta) = \int_0^\theta \frac{d\theta'}{\dot{\theta}(\theta')} = \frac{b}{v_{\max}} \int_0^\theta \sqrt{1+\theta'^2}\, d\theta' = \frac{b}{2v_{\max}} \left[\theta\sqrt{1+\theta^2} + \sinh^{-1}\theta\right].$$

This gives the angle as a function of time within this spiral : CLV. (Constant Linear Velocity) exists. To obtain the parametric equations for CLV, one must invert t to be a function of θ

4.2 Automated Tremor Assessment

Clinicians use the Archimedean spiral to quantify tremor severity. Patients draw a spiral modeled as

$$x(t) = b(t)\cos t + \delta_x(t), \quad y(t) = b(t)\sin t + \delta_y(t),$$

where $\delta_x(t), \delta_y(t)$ represent deviations caused by tremors. These deviations can often be modeled as

$$\delta(t) \approx A\cos(\omega t + \phi),$$

where amplitude A and frequency ω characterize the tremor. Machine learning models extract these features to analyze tremor severity.

4.3 Quadrature of the Circle

Archimedes attempted to construct a square with the same area as a given circle. Let the side of the square be s and the radius of the circle be r:

$$s^2 = \pi r^2 \quad \Longrightarrow \quad s = r\sqrt{\pi}.$$

Since $\sqrt{\pi}$ is not an algebraic number, this construction is impossible using classical tools.
Archimedes approximated the area using the Archimedean spiral:

$$\text{Area} = \frac{1}{2}\int_0^\theta r^2\,d\theta = \frac{1}{2}\int_0^\theta (b\theta)^2\,d\theta = \frac{1}{6}b^2\theta^3.$$

Setting this equal to the area of a circle:

$$\frac{1}{6}b^2\theta^3 = \pi r^2 \quad \Longrightarrow \quad \theta = \sqrt[3]{\frac{6\pi r^2}{b^2}}.$$

The area from the origin to this angle of an Archimedean Spiral is equal to the area of a circle with the radius r

4.4 Trisection of an Angle

The Archimedean spiral can be used to trisect an angle α. Let

$$r = a\theta \quad \Longrightarrow \quad \theta = \frac{r}{a}.$$

To trisect the angle, select r such that

$$\theta = \frac{\alpha}{3} \quad \Longrightarrow \quad r = a\frac{\alpha}{3}.$$

Thus, the corresponding angle along the spiral is

$$\theta = \frac{\alpha}{3},$$

completing the trisection.

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