Are All Infinities Equal?
Cantor's Theory of Infinite Cardinalities

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1 Hilbert's Hotel: A Paradox of Infinity

Imagine arriving late at night at a peculiar hotel with infinitely many rooms, numbered 1, 2, 3, 4, and so on. Every room is occupied, and yet, when you ask for a room, the manager smiles and says, "Of course, we have space."

With a single instruction, he asks each guest in room n to move to room n+1, instantly freeing up room 1 for you. A completely full hotel has made space for one more.

Before this paradox can even settle, an infinite number of new guests arrive. Once again, the manager remains unfazed. This time, he directs each guest in room n to move to room 2n, shifting all existing guests into even-numbered rooms. At once, every odd-numbered room, infinitely many of them, becomes available.

What seems impossible is, in fact, a defining feature of infinity. A set can be "full" and still admit more elements; it can even accommodate infinitely many more without growing in size.

This striking thought experiment, known as the Hilbert's Hotel paradox, forces us to confront a deeper question: if infinity can behave in such counterintuitive ways, how can we meaningfully compare the sizes of infinite collections?

2 Introduction

The notion of infinity has long occupied a central position in mathematics, philosophy, and logic. While early mathematicians treated infinity as an abstract and somewhat ambiguous concept, the development of set theory by Georg Cantor in the late nineteenth century provided a rigorous framework for its study. One of Cantor's most striking discoveries was that infinite sets need not all have the same size. In other words, there exist different "levels" or "magnitudes" of infinity.

The purpose of this work is to present a systematic and rigorous study of infinite sets and their cardinalities. We begin with fundamental definitions and gradually develop the theory leading to the hierarchy of infinite cardinals. Particular attention is given to key results such as the countability of rational numbers, the uncountability of real numbers, and Cantor's theorem on power sets.

3 Preliminaries and Notation

We begin by introducing the basic language and notation used throughout.

Let A and B be sets.

Definition 3.1 (Function)**.** A function f from A to B, denoted

$$f : A \to B,$$

is a rule that assigns to each element $a \in A$ a unique element $f(a) \in B$.

Definition 3.2 (Injective, Surjective, Bijective)**.** Let $f : A \to B$ be a function.

- The function f is said to be **injective** (one-to-one) if

$$f(a_1) = f(a_2) \implies a_1 = a_2 \quad \text{for all } a_1, a_2 \in A.$$

- The function f is said to be **surjective** (onto) if

$$\forall b \in B,\ \exists a \in A \text{ such that } f(a) = b.$$

- The function f is said to be **bijective** if it is both injective and surjective.

Definition 3.3 (Cardinality)**.** Two sets A and B are said to have the same **cardinality** if there exists a bijection

$$f : A \to B.$$

In this case, we write

$$|A| = |B|.$$

If there exists an injective function $f : A \to B$, we write

$$|A| \leq |B|.$$

Definition 3.4 (Countable and Uncountable Sets)**.** A set A is said to be **countable** if it is either finite or there exists a bijection

$$f : \mathbb{N} \to A.$$

If a set is not countable, it is said to be **uncountable**.

4 Countable Infinite Sets

We now study examples of infinite sets that are countable.

Theorem 4.1. The set of integers $\mathbb{Z}$ is countable.

Proof. We construct an explicit bijection between $\mathbb{N}$ and $\mathbb{Z}$.

Define $f : \mathbb{N} \to \mathbb{Z}$ by

$$f(n) = \begin{cases} \frac{n}{2}, & \text{if } n \text{ is even,} \\ -\frac{n-1}{2}, & \text{if } n \text{ is odd.} \end{cases}$$

We first show that f is injective. Suppose $f(n_1) = f(n_2)$. By considering the parity of n_1 and n_2, one verifies that this implies $n_1 = n_2$.

Next, we show that f is surjective. Let $z \in \mathbb{Z}$.

- If $z \geq 0$, then $z = f(2z)$.
- If $z < 0$, then $z = f(-2z + 1)$.

Thus every integer has a preimage in $\mathbb{N}$, and hence f is surjective.

Therefore, f is bijective, and $\mathbb{Z}$ is countable. □

$$|\mathbb{Z}| = |\mathbb{N}| = \aleph_0.$$

Theorem 4.2. The set of rational numbers $\mathbb{Q}$ is countable.

Proof. We first consider the set of positive rational numbers:

$$\mathbb{Q}^+ = \left\{ \frac{p}{q} : p, q \in \mathbb{N} \right\}.$$

Define a function

$$f : \mathbb{N} \times \mathbb{N} \to \mathbb{Q}^+, \quad f(p, q) = \frac{p}{q}.$$

We know that

$$|\mathbb{N} \times \mathbb{N}| = \aleph_0,$$

since pairs of natural numbers can be enumerated using a diagonal argument.

Now, the function f is surjective but not injective, since different pairs may represent the same rational number (e.g., $\frac{1}{2} = \frac{2}{4}$). However, removing duplicates from a countable set preserves countability.

Thus, $\mathbb{Q}^+$ is countable. Since

$$\mathbb{Q} = \mathbb{Q}^+ \cup \{0\} \cup (-\mathbb{Q}^+),$$

and a finite union of countable sets is countable, we conclude that

$$|\mathbb{Q}| = \aleph_0.$$

□

5 Uncountable Sets

Theorem 5.1 (Cantor's Theorem)**.** The interval $(0,1) \subset \mathbb{R}$ is uncountable.

Proof. Suppose, for contradiction, that $(0,1)$ is countable. Then there exists a sequence

$$x_1, x_2, x_3, \dots$$

containing all elements of $(0,1)$.

Write each x_n in decimal form:

$$x_n = 0.a_{n1}a_{n2}a_{n3}\dots$$

We construct a new number $y \in (0,1)$ as follows:

$$y = 0.b_1b_2b_3\dots$$

where

$$b_n = \begin{cases} 5, & \text{if } a_{nn} \neq 5, \\ 6, & \text{if } a_{nn} = 5. \end{cases}$$

By construction, $y \neq x_n$ for all n, since it differs in the n-th digit.

This contradicts the assumption that all real numbers in $(0,1)$ are listed.

Hence, $(0,1)$ is uncountable. □

6 Cantor's Theorem on Power Sets

Theorem 6.1. For any set A,

$$|A| < |\mathcal{P}(A)|.$$

Proof. Assume, for contradiction, that there exists a surjective function

$$f : A \to \mathcal{P}(A).$$

Define the subset

$$S = \{a \in A : a \notin f(a)\}.$$

Since $S \in \mathcal{P}(A)$, there exists $x \in A$ such that

$$f(x) = S.$$

Now we consider whether $x \in S$:

- If $x \in S$, then by definition $x \notin f(x) = S$, a contradiction.
- If $x \notin S$, then $x \in f(x) = S$, again a contradiction.

Thus, no such surjection exists, and hence

$$|A| < |\mathcal{P}(A)|.$$

□

7 Cantor–Bernstein Theorem

Theorem 7.1. If there exist injective functions

$$f : A \to B, \quad g : B \to A,$$

then

$$|A| = |B|.$$

Proof. (Outline) One constructs a bijection by partitioning A into disjoint subsets based on the behavior of f and g, and defining the bijection piecewise. This establishes a one-to-one correspondence between A and B. □

8 Aleph Numbers

Define:

$$\aleph_0 = |\mathbb{N}|.$$

Successive cardinals:

$$\aleph_1, \aleph_2, \ldots$$

form a strictly increasing sequence.

9 Ordinal Numbers

Ordinals extend counting beyond finite numbers:

$$\omega = \{0, 1, 2, \ldots\}.$$

Then:

$$\omega + 1,\ \omega + 2,\ \ldots$$

Ordinals encode order, unlike cardinals.

10 Continuum Hypothesis

$$2^{\aleph_0} = \aleph_1\ ?$$

This statement is independent of ZFC.

11 Hierarchy of Infinite Sets

$$\aleph_0 < 2^{\aleph_0} < 2^{2^{\aleph_0}} < \cdots$$

12 Countability of $\mathbb{N} \times \mathbb{N}$ (Detailed Proof)

Theorem 12.1.

$$|\mathbb{N} \times \mathbb{N}| = |\mathbb{N}|$$

Proof.

We prove that the Cartesian product $\mathbb{N} \times \mathbb{N}$ is countable by constructing an explicit bijection.

Define:

$$\mathbb{N} \times \mathbb{N} = \{(m, n) : m, n \in \mathbb{N}\}$$

We enumerate pairs along diagonals:

$$(1,1),(1,2),(2,1),(1,3),(2,2),(3,1),\ldots$$

Formally, define:

$$f(m,n) = \frac{(m+n-1)(m+n-2)}{2} + m$$

We show that f is bijective.

- **Injective:** Different pairs lie on different diagonals or different positions.
- **Surjective:** Every natural number corresponds to exactly one pair.

Hence:

$$|\mathbb{N} \times \mathbb{N}| = \aleph_0$$

13 Countable Union of Countable Sets

Theorem 13.1.
If $\{A_n\}_{n\in\mathbb{N}}$ is a countable collection of countable sets, then:

$$\bigcup_{n=1}^{\infty} A_n \text{ is countable.}$$

Proof.
Since each A_n is countable, there exists:

$$f_n : \mathbb{N} \to A_n$$

Define:

$$F : \mathbb{N} \times \mathbb{N} \to \bigcup_{n=1}^{\infty} A_n, \quad F(n,k) = f_n(k)$$

Since:

$$|\mathbb{N} \times \mathbb{N}| = \aleph_0$$

the image is countable.
Thus:

$$\left|\bigcup_{n=1}^{\infty} A_n\right| = \aleph_0$$

14 Uncountability of $\mathcal{P}(\mathbb{N})$

Theorem 14.1.

$$|\mathcal{P}(\mathbb{N})| = 2^{\aleph_0} > \aleph_0$$

Proof.
Apply Cantor's theorem:

$$|\mathbb{N}| < |\mathcal{P}(\mathbb{N})|$$

Also, each subset corresponds to a binary sequence:

$$A \subseteq \mathbb{N} \leftrightarrow (a_1, a_2, a_3, \dots), \quad a_i \in \{0, 1\}$$

Thus:

$$|\mathcal{P}(\mathbb{N})| = |\{0, 1\}^{\mathbb{N}}| = 2^{\aleph_0}$$

15 Real Numbers as Binary Sequences

Theorem 15.1.

$$|\mathbb{R}| = 2^{\aleph_0}$$

Proof Idea.
Each real number in $(0, 1)$ corresponds to a binary expansion:

$$x = \sum_{n=1}^{\infty} \frac{a_n}{2^n}, \quad a_n \in \{0, 1\}$$

Thus:

$$(0, 1) \sim \{0, 1\}^{\mathbb{N}}$$

Hence:

$$|\mathbb{R}| = 2^{\aleph_0}$$

16 Strict Growth of Power Sets

Theorem 16.1.

For any set A,

$$|A| < |\mathcal{P}(A)| < |\mathcal{P}(\mathcal{P}(A))| < \cdots$$

Proof

Repeated application of Cantor's theorem yields:

$$|A| < |\mathcal{P}(A)|$$

$$|\mathcal{P}(A)| < |\mathcal{P}(\mathcal{P}(A))|$$

Thus, an infinite strictly increasing chain exists.

17 The Schröder–Bernstein Theorem

While earlier sections outlined the idea, a formal statement of this theorem is essential for comparing cardinalities without constructing explicit bijections.

Theorem 17.1. If $|A| \leq |B|$ and $|B| \leq |A|$, then $|A| = |B|$.

The Injection f: There exists an injective function $f : A \to B$.

The Injection g: There exists an injective function $g : B \to A$.

The Partition: The set A can be partitioned into subsets based on chains formed by successive applications of f and g.

The Result: By defining a function piecewise on these subsets, one constructs a bijection between A and B, thereby proving $|A| = |B|$.

18 Properties of Cardinal Arithmetic

We now summarize important properties of infinite cardinal arithmetic, particularly for $\aleph_0$ and the continuum $c = 2^{\aleph_0}$.

- **Addition:**

$$\aleph_0 + \aleph_0 = \aleph_0$$

- **Multiplication:**

$$\aleph_0 \cdot \aleph_0 = \aleph_0$$

- **Power Sets:** For any set A,

$$|A| < |\mathcal{P}(A)|$$

- **Strict Growth:**

$$|A| < |\mathcal{P}(A)| < |\mathcal{P}(\mathcal{P}(A))| < \cdots$$

19 The Well-Ordering Principle

Definition 19.1. A set A is said to be *well-ordered* if every non-empty subset of A has a least element.

The well-ordering principle is fundamental in set theory and plays a key role in comparing cardinalities.

Ordinal vs. Cardinal: Cardinals measure the size of sets, while ordinals describe the order structure of elements. Examples of ordinals include:

$$\omega,\ \omega+1,\ \omega+2,\ \ldots$$

Succession of Cardinals: The sequence of aleph numbers

$$\aleph_0,\ \aleph_1,\ \aleph_2,\ \ldots$$

forms a strictly increasing hierarchy of infinite cardinalities.

20 The Continuum Hypothesis and Independence

Expanding on earlier discussion, we consider the gap between countable infinity and the continuum.

The Question:

$$\text{Does there exist a set } S \text{ such that } \aleph_0 < |S| < 2^{\aleph_0}\text{ ?}$$

ZFC Context: The statement

$$2^{\aleph_0} = \aleph_1$$

is independent of the Zermelo–Fraenkel axioms with the Axiom of Choice (ZFC).

Philosophical Impact: This result shows that infinity is not a single absolute concept but a structured hierarchy whose properties may depend on the chosen axiomatic system.

21 Summary of Infinite Hierarchy

Set	Description	Cardinality
$\mathbb{N}$	Natural Numbers (Countable)	$\aleph_0$
$\mathbb{Z}$	Integers (Countable)	$\aleph_0$
$\mathbb{Q}$	Rational Numbers (Countable)	$\aleph_0$
$(0,1)$	Unit Interval (Uncountable)	$2^{\aleph_0}$
$\mathbb{R}$	Real Numbers (Uncountable)	$2^{\aleph_0}$
$\mathcal{P}(\mathbb{N})$	Power set of $\mathbb{N}$	$2^{\aleph_0} > \aleph_0$

Table 1: Examples of Sets and Their Cardinalities

22 Philosophical Insight

Cantor's theory suggests:

- Infinity is not absolute
- Mathematics allows comparison of infinite magnitudes
- There is no "largest infinity"

23 Conclusion

In this work, we developed the theory of infinite sets and cardinalities, starting with the distinction between countable and uncountable sets. We established that familiar sets such as $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{Q}$ are countable, while the set of real numbers is uncountable, proving that not all infinities are equal. Using Cantor's theorem, we showed the power set of any set has greater cardinality, creating an infinite, increasing hierarchy of infinities.

Beyond these foundational results, we also explored structural aspects of set theory, including ordinal numbers (capturing order) and the Continuum Hypothesis, highlighting limits of axiomatic systems. These show infinite cardinalities are not isolated but part of a structured framework governed by precise principles.

Thus, infinity is not singular but a vast, intricate hierarchy expanding through successive constructions. Cantor's insights revolutionized mathematics and reshaped understanding of size, order and mathematical truth. Infinite sets remain central to modern mathematics, influencing topology, analysis, logic and inspiring further study.

This exploration has been particularly fascinating, as it challenges some of our most fundamental intuitions about size and quantity. What begins as a simple question about infinity quickly unfolds into a rich and intricate theory, connecting ideas from set theory, logic, and beyond. I thank you, the reader, for reading my essay and I hope you enjoyed this journey as much as I have!