Banach-Tarski Paradox

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INTRODUCTION:

Imagine holding a ball in your hand. Now we slice this ball into 5 carefully chosen pieces. Would you believe me if I told you that you could now arrange these pieces to form two identical copies of the original. Not smaller, not distorted, the **same.** It begs the question; how can something be duplicated without adding extra material? Well, in 1924 two Mathematicians by the names, Stefan Banach and Alfred Tarski discovered this.

This paradox does not mean that we can create matter, rather it exposes something deeper: the consequences of infinity and the limits of measurements.

ASSUMPTIONS OF VOLUME:

In classic Geometry, volume behaves as expected. If a solid cube is dividing into pieces, the volume remains unchanged. This principle relies on the assumptions of additivity: the idea that volume of a union of disjoint sets equals the sum of the individual volumes.

$$\mu(A \cup B) = \mu(A) + \mu(B)$$

Where μ denotes the volume. If we extend this idea to infinitely many sets, we get the following:

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i)$$

This framework is formalised through Lebesgue measure, which is the standard way to assign 'length' to subsets of Euclidean space. However, not all sets have a well-defined volume. Some subsets cannot have a volume assigned to them; they are known as unmeasurable sets. The key point is: if a set can exist that cannot be measured consistently, then rearranging can lead to an unexpected result.

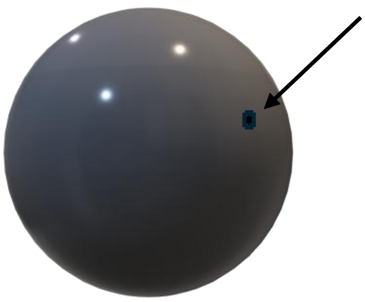
THE HYPERWEBSTER:

The idea of the "Hyperwebster" was introduced by mathematician, Ian Stewart, which can be used to help explain the strange ideas behind this paradox. It consists of a theoretical dictionary which lists every possible combination of letters within the 26-letter alphabet – common sense tells us that this in infinity large. Volume A would look something like this:

A, AA, AAA, AAAA…
AAB, AABA, AABAA… →AZ, AZA, AZAA, AZAAA…

What's so fascinating about this is that is we take Volume A and remove the first 'A' we are actually left with every possible combination of letters. The entire 26 volumes will be condensed down into one. We can dive into this paradox by truing our initial sphere into a hyperwebster.

SPLITTING UP OUR SPHERE:



First take a starting point for our sphere. We need to cover the whole surface. The best way to do this is by moving along the surface of the sphere in steps with the measurement of arccos(1/3), either left, right, up or down. As long as we don't backtrack no point will be covered more than once (aside from the poles). If all possible routes are taken along the surface of the sphere we will have an infinite set of points. We then will colour code each point with the last rotation. For example, if I have the sequence RUL (right, up, left), this would be coded with the colour associated with right. If supposedly we missed any points, we could just relocate the starting point here and run all of the permutations again.
At the end you will be left with infinite points of, up, down, left, right, starting points and poles lines which stem to the centre.

What the UP-DOWN-LEFT-RIGHT set of points look like:

U,UU,UUU,UUUU…	D,DD,DDD,DDDD…	L,LL,LLL,LLLL…	R,RR,RRR,RRRR…
UL,ULL,ULLL…	DL,DLL,DLLL…	LU,LUU,LUUU…	RU,RUU,RUUU…
UR,URR,URRR…	DR,DRR,DRRR…	LD,LDD,LDDD...	RD,RDD,RDDD…

After all this we are left with 6 pieces of sphere. Up , Down , Left , Right , Pole and Starting point lines.

ROTATING OUR PIECES AND ASSEMBLING:

If we take a look at the Left lines and we rotate it right we could add an 'R' to the list of rotations. However, if we look closely 'RL' is the same as no rotation at all. We are left with all the up points, down points, left points and starting points by simply rotating the structure right. ¼ of a sphere all the way to ¾! If we add the right points, and poles we are left with a whole sphere…but with pieces left over (the down points, starting points, the up points) to make another. Similarly, with some crafty rotation we can assemble the remaining pieces into another sphere, which is missing the poles points. To fill these missing points in we need to look at the Infinite Hotel Paradox.

INFINITE HOTEL:

Suppose we have N rooms, where N in infinite. 1,2,3,4…N. When a new guest arrives, we can just all guests one room to the right. Which leaves an empty room for the new guest. The same logic can be applied to the missing pole points, we can fill them it by shifting the points next to them into their spot. Then the points next to the points that just moved will move and so on. In theory all points will be filled. This leaves us with 2 identical spheres.

2D?

This paradox only works in 3 dimensions:
The Banach–Tarski paradox does not work in 2D because the group of rotations in the plane is abelian, meaning rotations add up: for any angles $\theta_1$and θ_2,

$$R_{\theta_1}R_{\theta_2} = R_{\theta_2}R_{\theta_1}$$

This simple structure makes the group amenable. So it allows additive, rotation , which prevents paradoxical decompositions of shapes like circles. In contrast, in 3D the rotations are non-abelian, so in general:

$$R_x(\alpha)R_y(\beta) \neq R_y(\beta)R_x(\alpha)$$

TO CONCLUDE:

The Banach Tarski paradox is one of the interesting paradox's in math. Rather than contradicting geometry it exposes it. More importantly it lets mathematicians approach infinity in a different way. Volume appears to behave differently when applied to non-measurable sets. Obviously, attempting this in real life would not work but it defines the border between theoretically possible and infinity.

REFERENCES

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Thanks for reading.