The Banach-Tarski Paradox: Where Geometry Challenges Intuition

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1 Introduction

To quote from Parmenides, "ex nihilo nihil fit" meaning "nothing comes from nothing," but, what if it can? Seemingly counterintuitive, Polish mathematicians Stefan Banach and Alfred Tarski proposed the Banach-Tarski paradox which defies mathematical intuition to yield an implausible result. It proves that there is in fact a way to take an object, and separate it into 5 pieces. And then, with those 5 pieces, simply rearrange them into 2 exact copies of the original. Widely considered one of the most counterintuitive results in mathematics, it challenges the fundamental idea that you cannot make something from nothing, which raises many questions.

2 Consequences of infinity

Not all infinities are equal

First, what *is* infinity? You might be able to give the Oxford definition where it is "a number greater than any assignable quantity or countable number," but to tackle Banach-Tarski we need to set more boundaries. Infinity might seem like a number, especially through the saying of "an infinite number of..." (which is in fact incorrect.) Instead, it is rather a size; the size of something that doesn't end. It is not the biggest number, but how many numbers there are. Importantly, there are different sizes of infinity; countable and uncountable. The former of the two is the smallest, and can be represented by the amount of natural numbers. Sets like these are unending, but are countable.

Uncountable infinity, on the other hand, is too large to count. This can be the amount of real numbers in the universe, or just the numbers between 0-1 as shown on Figure 1.

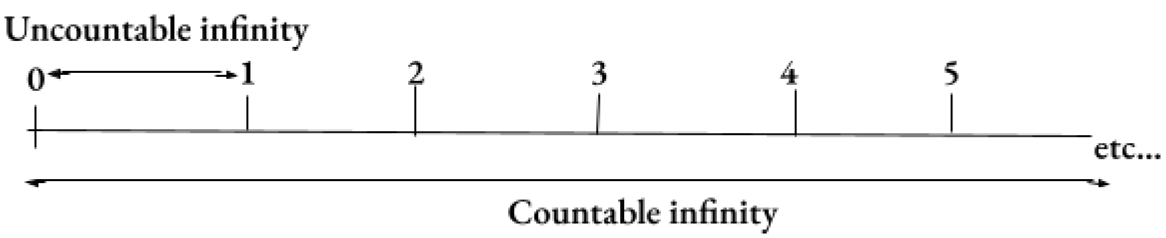


Figure 1: The uncountable infinity 0-1 compared to the number of natural numbers (a countable infinity.)

Hilbert's Hotel

Hilbert's Hotel is a concept which states that an infinite hotel, even when completely occupied with an infinite number of guests, can always keep more guests. It shows that, unlike finite sets, infinite sets do not increase in size when numbers are added, highlighting that "infinity plus one" still equals infinity.

Imagine this hotel, where there is a countably infinite number of rooms, and all rooms are occupied. So, it is fully booked, right? Let's say one guest appears and asks for a room. It may seem difficult to accommodate at first, but what we can do is just shift each guest over one room, so whoever was in room 1 goes to room 2, room 2 to room 3, etc. Now, the guest who has just arrived is in room 1 and all rooms are still occupied. Note that since the number of rooms is never ending, you can never run out of rooms, but they are also always occupied.

We can also apply this when going backwards, to simulate subtracting 1 from infinity. Imagine 1 guest leaves the hotel, we can again just shift each guest, this time going the other way. So, guest 4 to room 3, guest 3 to room 2, etc. This highlights the properties of infinity, where you can add or subtract any finite number yet still obtain infinity.

Hilbert's Hotel on a circle

Importantly, we can apply Hilbert's Hotel to a circle, which will prove crucial for the Banach-Tarski paradox. Instead of guests in a hotel, we can imagine them as points on the circumference. Now, if we remove one point from the circle, it's gone, right? Infinity argues otherwise. We can use the principles found from Hilbert's Hotel to prove the circle is still complete. Let each point on the circle be labelled moving clockwise with a natural number, such that the length of the radius is the distance between each point. Note that since the circumference of a circle is irrational, being $2\pi r$, no point can ever be labelled with the same number. This set shown in Figure 2 is an example of a countable infinity, similar to guests in Hilbert's Hotel. As we shifted the guests once one checked

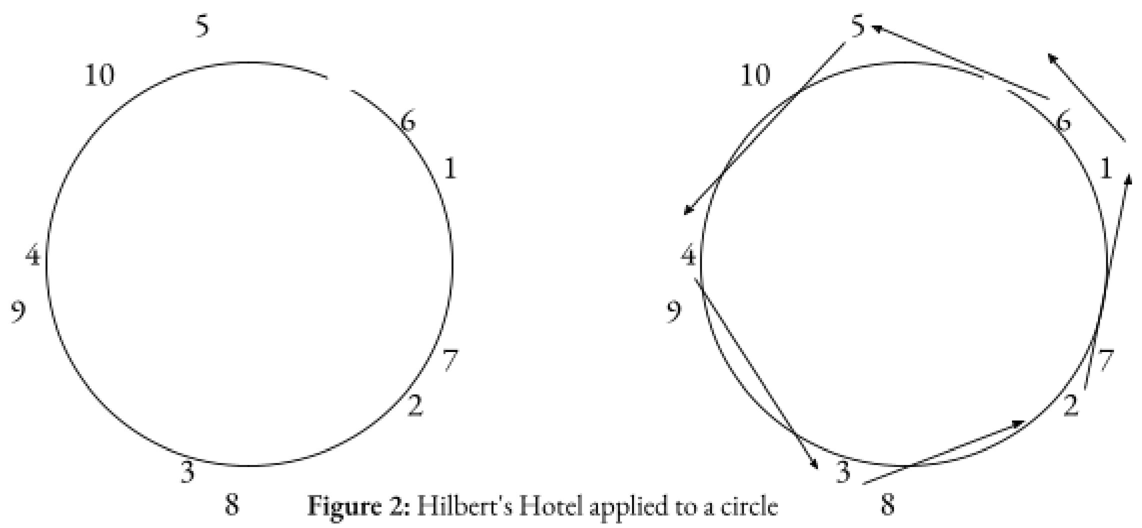


Figure 2: Hilbert's Hotel applied to a circle

out, we can shift points anticlockwise once the point was removed, so point 1 moves to the gap, point 2 to point 1 etc until all points are filled. Since there is an infinite supply of points, they will never run out and the initial gap will be forgotten. It was never necessary to be complete.

Hyperwebster

Ian Stewart famously proposed a brilliant dictionary. One that he called the Hyperwebster. This would contain every single possible word that could be created using the 26 letters in the English alphabet. The infinite number of words could be collected such that we have:

Volume 1: A,AA,AAA…,AB,ABA,ABAA…,AC,…,AZ,AZA,…
Volume 2: B,BA,BAA…,BB,BBA,BBAA…,BC,…,BZ,BZB,…
.
.
.
Volume 26: Z,ZA,ZAA…,ZZ,ZZA,ZZAA…,ZC,…,ZZ,ZZZ,…

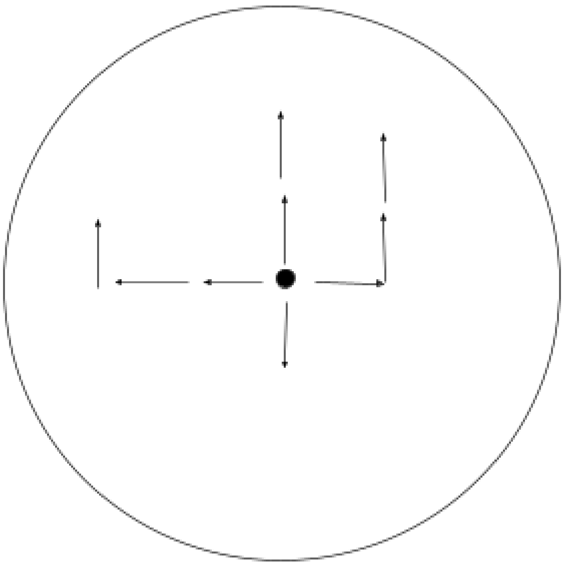
Next, we can name each volume with the letter it begins with, to avoid having to write it out each time. Then we form these following volumes:

Volume 1:A,AA,AAA,…,B,BA,BAA,…C,…,Z,ZA,…
Volume 2:A,AA,AAA,…,B,BA,BAA,…C,…,Z,ZA,…
.
.
.
Volume 26:A,AA,AAA,…,B,BA,BAA,…C,…,Z,ZA,…

Interestingly, we can observe that all volumes now contain the same information, except for the volume name. Therefore, publishing a single volume would now contain all possible words and can be done instead. This later introduces the possibility of creating a 3D shape that can be decomposed into multiple parts, vital for the explanation of the Banach-Tarski paradox.

3 The main proof

To finally begin to prove the Banch-Tarski paradox, we can combine aspects of the previous ideas discussed above. Firstly, we can name each point on a sphere with the directions used to reach it from a given starting point. By going across the 2 axes, we can use the rotations: left, right, up, and down. Importantly, we cannot "backtrack" by repeating two countering steps after each other, for example "left right." Some possible directions can be shown through Figure 3, and every single possible sequence that can be made of any finite length out of just these four rotations will be needed. The rotations are in order right to left, so the final rotation is the leftmost letter, which will be of importance later.



Possible directions:
L
R
U
D
LL
LR
LU
LD
etc.

Figure 3: diagram of sphere and possible rotations to travel across it

Considering every single sequence of rotations will result in a countably infinite number. Through Figure 4, we can see that if we apply each one to the starting point, labelled in red, and then name the point we land on after the sequence that brought us there, we can name a countably infinite set of points on the surface. The examples LDR, UU, and R are shown on the diagram. For example, in the sequence LDR, we end rotating right, so have coloured it purple. Completing this process for all possible sequences would result in a countably infinite result, although not what is needed.

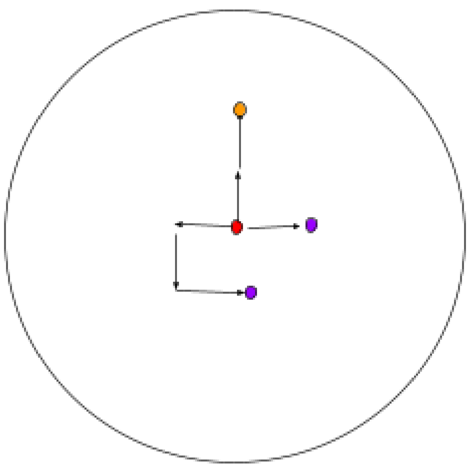


Figure 4: final directions from the sphere colour coded.

Key:

Starting point = red

Left = green

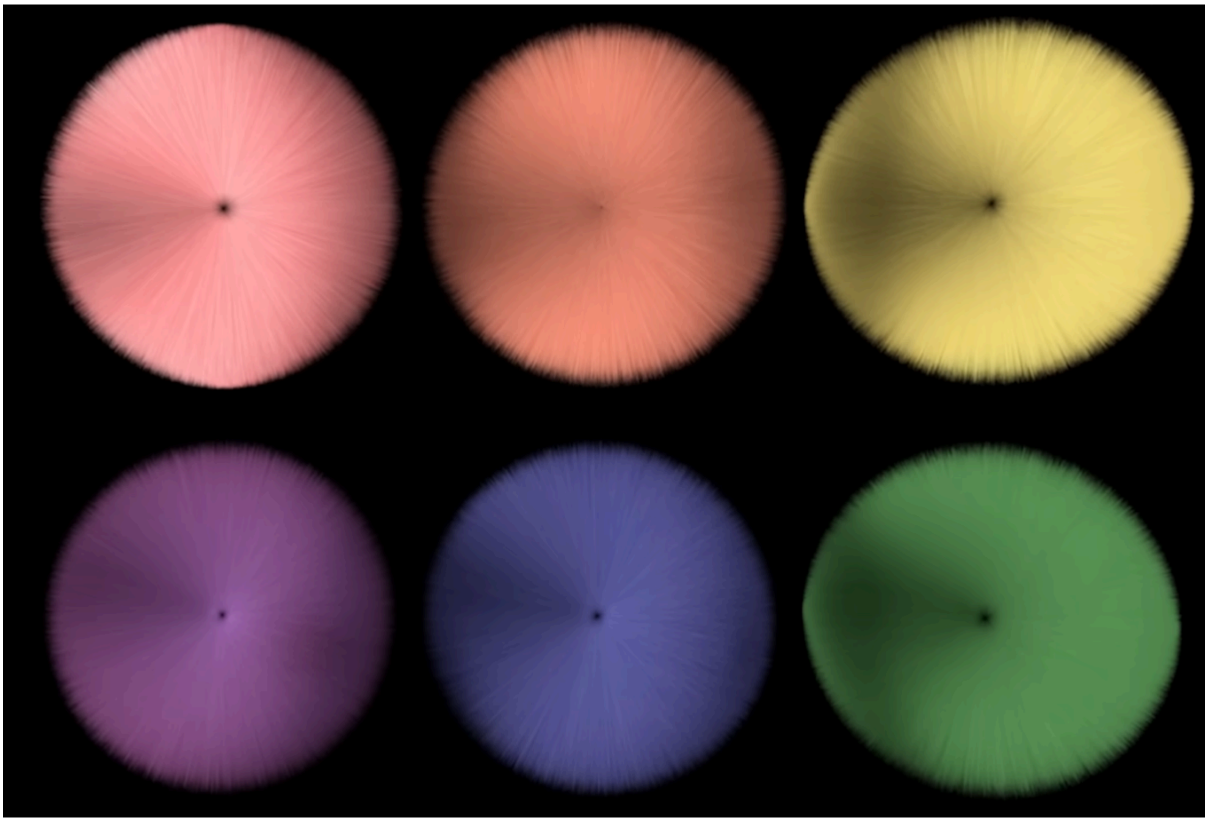
Right = purple

Up = orange

Down = blue

Poles = yellow

Due to the sphere's irrational and infinitely long surface area, we need an uncountably infinite number of points to cover the entire surface. To do this, we can imagine finding every unaccounted for point and colouring it red to signify a new starting point, and run every sequence from there. After doing this to an uncountably infinite number of starting points, we will have named and colored every single point on the surface just once. The only exception is poles, located in the north and south, because more than one sequence can lead us to them; they can be named more than once and be colored in more than one color. For example, if you follow some other sequence to the north or south pole, any subsequent rights or lefts will be equally valid names. In order to counter this, we can label the poles in yellow and exclude them from the scheme, and since every scheme will have 2 poles, there is a countably infinite amount. Now, every point on the sphere has been accounted for and labelled with 1 of 6 colours, so we can imagine separating the sphere into 6 groups of coloured points, as shown in the image below.



Clearly, these points could be joined together to form one copy of the original sphere, although there is a way to create 2. First, we can notice that the left piece, labelled in green, is composed of every point, accessed via a sequence ending with a left rotation. So, if this piece is rotated to the right, this is equivalent to adding an "R" to the start of the sequence (reading right to left.)

Although, this is "backtracking" which was not permitted, allowing the R and L to cancel each other out. Now when they are reduced away, this has, similar to the Hyperwebster, created almost the entire sphere, by adding nothing. We can add in the missing poles of rotation in yellow and starting point in red to achieve an exact copy of the starting sphere, with pieces left over.

To create a second copy, we can similarly take the up piece and rotate it down. This again will result in a backtrack, so now it contains right, left, up pieces and starting points. However, this creates an error; we do not need the extra starting points since we have not used the first yet. Instead, we can just move everything from the up piece that turns into a starting point when rotated down. That means every point whose final rotation is up. Then we can realise, we need to move all points with any name that is just a string of Us. We will put them in the down piece and rotate the up piece down, which makes it congruent to the up right and left pieces, add in the down piece along with the up and the starting point piece. Finally, the poles of rotation and center are missing from this copy, but since there is a countably infinite number of holes, where the poles of rotations used to be, we can see this is just many circles with one point missing. Following ideas from Hilbert's Hotel when on a circle, we can fill those holes in. Now, the second sphere has been formed. We have proven that one sphere can make two; 1 + 1 (can) = 1.*

*This only works regarding infinitely many points.

4 Conclusion

We have taken one sphere and turned it into two identical spheres without adding anything. One plus one equals 1. The implications are puzzling, and mathematicians, scientists and philosophers are still debating them. Could such a process happen in the real world? We still don't know. History is full of examples of mathematical concepts developed in the abstract that we did not think would ever apply to the real world for decades (like RSA encryption involving primes,) until they were eventually found useful and applicable. For the Banach-Tarski Paradox to occur in real life, we would need infinitely complex and detailed measurements, although this is incredibly difficult considering the finite time to do anything. There have been a number of papers published suggesting a link between by Banach-Tarski and the way sub-atomic particles can collide at high energies and turn into more particles than we began with. We are finite creatures. Our lives are small and can only scientifically consider a small part of reality. What's common for us is just a sliver of what's available. We can only see so much of the electromagnetic spectrum; we can only delve so deep into extensions of space. Common sense applies to that which we can access.

But common sense is just that. Common. If total sense is what we want, we should be prepared to accept that we shouldn't call infinity unusual. The results we've arrived at by accepting it are valid, true within the system we use to understand, measure, predict and order the universe. Perhaps the system still needs perfecting, but at the end of day, history continues to show us that the universe isn't strange. We are.