The Forgotten Formula – An Exploration into the Cubic Formula

1. The Introduction

I'm in year ten and I finally felt like I cracked something monumental: The Quadratic Formula. A single line which could solve any equation with x^2 in it. But a question was nagging me from the back of my mind – what if I went one step further? Was it even possible? A formula just like the quadratic, just plugging in numbers into an equation. My curiosity led me to find what seemed to be an abomination! But instead of ignoring it and moving on, I wanted to understand it – no I wanted to conquer it!

So here we are. This essay is my journey from $ax^2 + bx + c$ to the quadratic formula, then using that as a basis to attempt to work out how to form the cubic formula from scratch. We will investigate how this formula can link to graphs and how us as curious humans, discovered it in the first place!

2. How to derive the Quadratic Formula

To understand how we got to the cubic formula, we fist need to understand the basics, by knowing how we even got to the quadratic formula. Let's take the general equation:

$$ax^2 + bx + c = 0$$

We want to get x by itself so we can start by moving the constant (c) onto the other side of the equation. We can also divide everything by the coefficient of a to get:

$$x^2 + \frac{b}{a}x = -\frac{c}{a}$$

Now we get to the critical part. We want to turn this expression into a perfect square trinomial which will basically "collapse" an expression and rewrite it into an expression with one x term. This is done by completing the square – we add the specific value (half the middle coefficient, squared) to each side.

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2$$

Therefore:

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{4ac}{4a^2}$$

Hmm... It's easy to understand ...

For now...

On the right side the denominators are the same so we can simplify further so it is one fraction. From this, we can solve x. Start by square rooting both sides.

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$$

Of which the $4a^2$ can be simplified into 2a. All you need to do now is $-\frac{b}{2a}$ from each side to isolate x

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Now you have successfully recreated the discovery of the quadratic formula!

3. Tackling the Cubic Formula

This formula is named after Gerolamo Cardano who published the general, algebraic solution in his 1545 book. He was allegedly the first to discover it (more on that later…) Anyways, here is the cubic formula:

$$x = \sqrt[3]{-\frac{q}{2}+\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}}+\sqrt[3]{-\frac{q}{2}-\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}}-\frac{b}{3a} \;\; \textbf{\textit{where}} \;\; p=\frac{3ac-b^2}{3a^2} \;\; \textbf{\textit{and}} \;\; q=\frac{2b^3-9abc+27a^2d}{27a^3}$$

Oh sorry. I forgot to mention – that was the simplified version! The full Cardano formula is:

$$x = \sqrt[3]{-\frac{b^3}{27a^3}+\frac{bc}{6a^2}-\frac{d}{2a}+\sqrt{\left(-\frac{b^3}{27a^3}+\frac{bc}{6a^2}-\frac{d}{2a}\right)^2+\left(\frac{c}{3a}-\frac{b^2}{9a^2}\right)^3}}$$
$$+\sqrt[3]{-\frac{b^3}{27a^3}+\frac{bc}{6a^2}-\frac{d}{2a}-\sqrt{\left(-\frac{b^3}{27a^3}+\frac{bc}{6a^2}-\frac{d}{2a}\right)^2+\left(\frac{c}{3a}-\frac{b^2}{9a^2}\right)^3}}-\frac{b}{3a}$$

But how did we get from here from ax^3+bx^2+cx+d? Let's break this down into steps. Similar to the quadratic formula, we divide everything by "a" to make the expression monic (the biggest term's coefficient = 1). This gives: $x^3+\frac{b}{a}x^2+\frac{c}{a}x+\frac{d}{a}=0$. To make things easier. We can make $\frac{b}{a}=A$, $\frac{c}{a}=B$, and $\frac{d}{a}=C$. We now have the equation:

$$x^3+Ax^2+Bx+C=0$$

Thinking about the shape of the graph, a cubic has a point of inflection (the point where the curve changes direction). If we shift this cubic function on a graph, so that this point lies on the y-axis, the x^2 term becomes 0 (see section 4 **Graphs**). This is known as the Vieta Transformation.

Talking about the x^2, we want to eliminate it. Here comes the tricky part: Let $x=y-h$. We do this because x is the original unknown, and y would be the new unknown after moving the point of inflection to be aligned with the y axis. The "h" is their distance relation. Now we substitute x:

$$(y-h)^3+A(y-h)^2+B(y-h)+C=0$$

By expanding all of the brackets, you get a long, string of terms:

$$(y^3-3hy^2+3h^2y-h^3)+(Ay^2-2Ahy+Ah^2)+(By-Bh)+C=0$$

To make the equation more sensical, we can group all the like terms together (e.g. constants, y^3…):

$$y^3+(-3h+A)y^2+(3h^2-2Ah+B)y+(-h^3+Ah^2-Bh+C)=0$$

Due to the Vieta Transformation, we know $y^2=0$ which means $-3h+A=0$. From here we can work out that $h=\frac{A}{3}$ which is critical! We can now get rid of the x^2 term and substitute in h, giving us:

$$y^3+\left(3\left(\frac{A}{3}\right)^2-2A\left(\frac{A}{3}\right)+B\right)y+\left(-\left(\frac{A}{3}\right)^3+A\left(\frac{A}{3}\right)^2-B\left(\frac{A}{3}\right)+C\right)$$
$$=y^3+\left(B-\frac{A^2}{3}\right)y+\left(\frac{2A^3}{27}-\frac{AB}{3}+C\right)=0$$

Finally for more simplification, let $\boldsymbol{p} = B - \frac{A^2}{3}$ and $\boldsymbol{q} = \frac{2A^3}{27} - \frac{AB}{3} + C$. Now we have a simple looking formula of $y^3 + py + q = 0$. This expression is known as the depressed cubic.

If we think back to the curve, each point is planted along the weird looking squiggle. This squiggle has 2 parameters – steepness/wiggle in the curve (p) and vertical position (q). Therefore, we can make 2 new variables equal to y: $y = u + v$. This is to make the solving easier as its split into two equations and later on, u^3 and v^3 are 2 roots of a quadratic equation, but more on that later.

Now that $y = u + v$, we can plug this into our expression to get:

$$(u+v)^3 + p(u+v) + q = 0$$

Once again, we can use binomial expansion to rearrange the terms into:

$$(u^3 + v^3 + 3u^2v + 3uv^2) + p(u+v) + q = 0$$

With some smart factoring, we can factor out $3uv$ from the middle terms and get:

$$(u^3 + v^3 + q) + (\boldsymbol{u} + \boldsymbol{v})(\boldsymbol{3uv} + \boldsymbol{p}) = 0$$

Looking specifically at the bold section, we can make $3uv + p = 0$ because we made up u and v to make the y from earlier. Now we can discover two clear rules:

1. $u^3 + v^3 = -q$
2. $uv = \frac{-p}{3}$ therefore $u^3v^3 = \frac{p^3}{27}$

This shows that u and v are the roots of a hidden quadratic! – We know the sum and products of 2 numbers an her is proof! Start with a normal quadratic equation:

$$(z - u^3)(z - v^3) = 0$$

In this case, z is another "filler" varaiable. Then we can expand the brackets and group like terms by z:

$$z^2 - v^3z - u^3z + u^3v^3 = 0$$

$$z^2 - (u^3 + v^3)z + u^3v^3 = 0$$

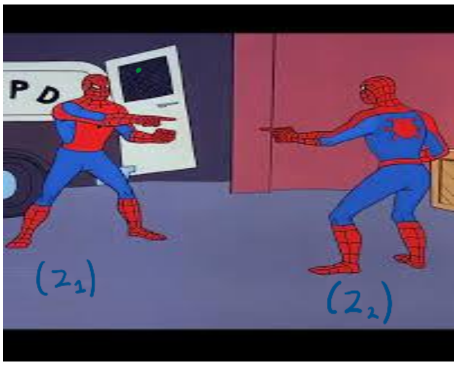
Notice how we can see the sum and the product in the equation. Now we substitute $-q$ and $\frac{p^3}{27}$ into the quadratic:

$$z^2 - (-q)z + \left(-\frac{p^3}{27}\right) = z^2 + qz - \frac{p^3}{27} = 0$$

Now we have to work out the values for z. Each different value is actually u^3 *and* v^3! After using the quadratic formula from earlier, we figure out z to be:

$$\boldsymbol{z1} = -\frac{\boldsymbol{q}}{\boldsymbol{2}} + \sqrt{\frac{\boldsymbol{q}^2}{\boldsymbol{4}} + \frac{\boldsymbol{p}^3}{\boldsymbol{27}}}$$

$$\boldsymbol{z2} = -\frac{\boldsymbol{q}}{\boldsymbol{2}} - \sqrt{\frac{\boldsymbol{q}^2}{\boldsymbol{4}} + \frac{\boldsymbol{p}^3}{\boldsymbol{27}}}$$



Z_1: I'm z!

Z_2: No I'm z!

Z_3: (Non-existent as its a quadratic)

It does not matter which value is u and v, as long as you remember to cube root both answers. In terms of y, we have:

$y = \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$ and remember, $x = y - h$ and we worked out h to be $\frac{b}{3a}$

Now we have the simplified cubic formula:

$$x = \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} - \frac{b}{3a}$$

Now we can substitute the q and p values $\left[p = \frac{3ac - b^2}{3a^2} \ \boldsymbol{and} \ q = \frac{2b^3 - 9abc + 27a^2d}{27a^3}\right]$. Dividing q by -2 gives us:

$$-\frac{\boldsymbol{q}}{\boldsymbol{2}} = \left(\frac{2b^3}{54a^3} - \frac{9abc}{54a^3} + \frac{27a^2d}{54a^3}\right) = -\frac{b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a}$$

We can also work out:

$$\frac{\boldsymbol{p}}{\boldsymbol{3}} = \frac{3ac}{9a^2} - \frac{b^2}{9a^2} = \frac{c}{3a} - \frac{b^2}{9a^2}$$

Using these critical values, we can now rearange the cubic formula into a simpler form:

$$x = \sqrt[3]{-\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}} + \sqrt[3]{\left(-\frac{q}{2}\right) - \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}} - \frac{b}{3a}$$

Finally, when putting in the values, we get:

$$x = \sqrt[3]{-\frac{b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a} + \sqrt{\left(-\frac{b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a}\right)^2 + \left(\frac{c}{3a} - \frac{b^2}{9a^2}\right)^3}}$$
$$+ \sqrt[3]{-\frac{b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a} - \sqrt{\left(-\frac{b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a}\right)^2 + \left(\frac{c}{3a} - \frac{b^2}{9a^2}\right)^3}} - \frac{b}{3a}$$

So that’s great! We have our formula. BUT, in the world of complex numbers (e.g. “i”), every number has exactly three cube roots. So how do we get three solutions for x with one formula?

All we have to do, is find the first, “normal” cube root of u. We cannot use v because of one of our rules from earlier translate to $-3uv = -p$. As soon as you select your “u” term, the v term is automatically decided for you. So the three solutions are the main cube root of u, times by $\omega^0, \omega^1, \omega^2$. The Greek symbol, omega, are the cube roots of unity, which are the three, complex solutions of $n^3 = 1$.

We can work out the roots of unity by solving the quadratic of $n^2 + n + 1 = 0$ (This equation is formed from $n^3 = 1$ and algebraically rearranging and factoring it. Anything to the power of 0 = 1 so:

$$\omega^0 = 1$$

$$\omega^1 = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$$

$$\omega^2 = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$$

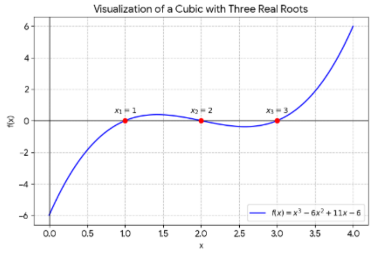
$$\omega^3 = 1$$

If you didn't know already, "i" is the imaginary number of the $\sqrt{-1}$. Now due to our rule from earlier, the combinations of omegas and u/v are specific, which are:

$$u + v$$

$$\omega u + \omega^2 v$$

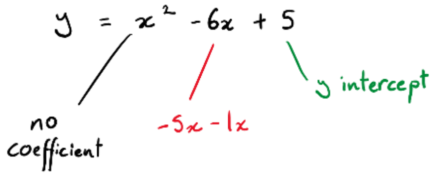
$$\omega^2 u + \omega v$$

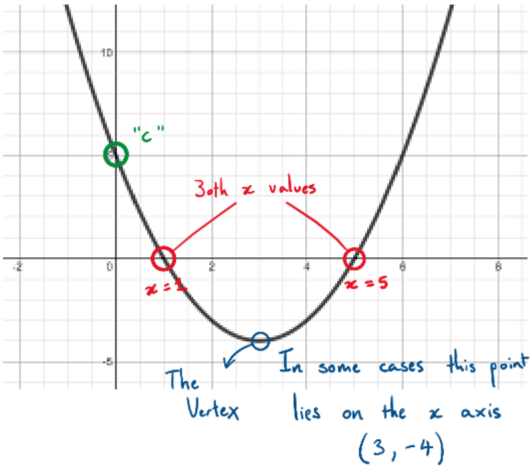


The first expression is our standard pair and the other two make sense because the product of omegas add up to one ($\omega^2 + \omega = \omega^3 = 1$). Now we have a way to get three solutions and **have mastered the cubic formula…** Wait – let's not forget graphs!

4. The Graphs

In the quadratic formula, the two x values are the exact, x intercepts of the parabola shape, with the constant (c) indicating the y intercept.





Therefore, we can extract information from the cubic formula that can be presented on a cubic graph. Like the quadratic formula, the constant term (d) represents the y intercept. Similarly, each of the root solutions are the 3 x-intercepts. It will not always be 3 real roots, as we have imaginary, complex roots but they will always be 3.

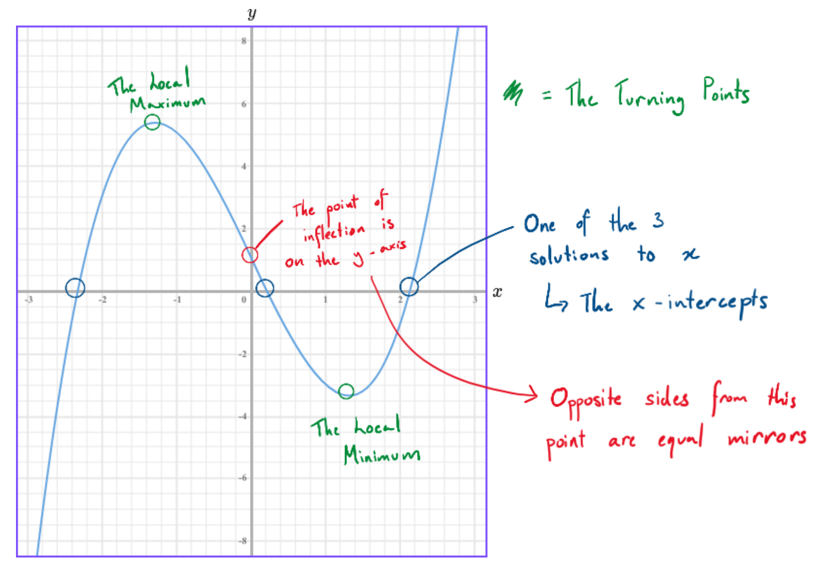
If a > 0, the graph will go from bottom-left to top-right whereas if a < 0, the graph line will go from top-left to bottom-right. Do note, a cannot be 0 or it will be a simple, quadratic equation. As mentioned earlier, the inflection point of the curve is when $x = -\frac{b}{3a}$.

If $\Delta > 0$: 3 x-intercepts

If $\Delta = 0$: 2 or less x-intercepts

If $\Delta < 0$: exactly 1 x-intercept

$a > 0$



$y = x^3 - 7x + 1$ (in this graph there is no x^2)

Notice how I used the symbol Δ. This represents the discriminant which in a graph, shows the nature of the roots – it shows how many times the curve crosses the x axis, which is the number of real roots. The discriminant in a quadratic formula is just the $b^2 - 4ac$. For the cubic formula, the discriminant is:

$$\Delta = 18abcd - 4b^3 + b^2c^2 - 4ac^3 - 27a^2d^2$$

Using graphs help prove the cubic formula as certain co-ordinates of significant points, such as the point of inflection, should aline with the answers to the corresponding expression.

5. The Scandalous History of the Lost Formula

The story of the cubic formula is a tale of secrecy, rivalry, and a race to unlock the mysteries of numbers - almost like a mathematical thriller. Its roots trace back to the Renaissance, a time when Europe was bursting with new ideas, and mathematicians were pushing the boundaries of what numbers could do.

As I mentioned earlier, a man named Cardano “constructed” this gargantuan formula but their were others before him! A professor at the University of Bologna, called Scipione del Ferro, reduced the cubic equation to the depressed cubic (a cubic equation without an x^2 term). He managed to work out one of the roots obviously he didn’t know about the three roots of unity…

In 1535, a self-taught mathematician – Tartaglia – discovered how to solve a cubic equation with a missing linear term. Cardano at the time, was frustrated and made a pressure campaign to convince Tartaglia to share his method. Part of his vow states: *“I swear to you by the sacred gospel… never to publish your discoveries”*.

But promises in the Renaissance were fragile. Cardano, ever ambitious, eventually discovered that del Ferro had solved similar cubic equations years earlier. Feeling justified, he went ahead and published Tartaglia’s solution in his Ars Magna in 1545, giving minimal credit to del Ferro as the originator. This bold move made waves in the mathematical world - Tartaglia felt betrayed, and the story of secrecy and rivalry only grew more dramatic.

But Cardano didn't stop there. With the help of his brilliant student, **Lodovico Ferrari**, he tackled the even more complicated quartic equations (degree four). Together, they transformed algebra from a puzzle-solving tool into a fully systematic science. The cubic formula, with all its twists and turns, also led mathematicians to face strange new numbers (what we now call complex numbers) long before anyone fully understood them.

6. The Conclusion

Well there it all is. The story of the cubic formula reminds us that mathematics is more than just a set of rules or symbols. Maths is a tool which has the power to reveal patterns, connections and beauty in a world that feels as chaotic. As tumultuous as this formula may have appeared, it is simply human curiosity that allowed us to understand such fun, abstract phenomenons.

The cubic formula is more than a solution to an equation—it is a story of curiosity, courage, and creativity. It shows that math is alive, full of patterns waiting to be discovered, and that even the most abstract ideas can reveal unexpected beauty. In numbers, we find not just answers, but wonder.

Thank you for reading – this was a really fun project.

By Calvin Yau