Can Fractals Really Be Frozen?

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Introduction

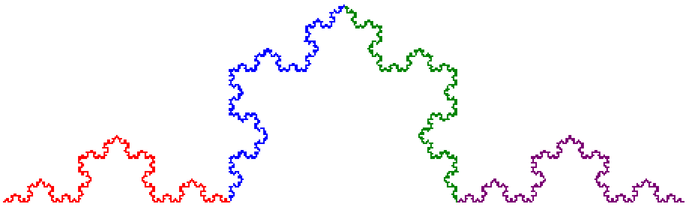
As a child, some of my fondest memories involve watching the movie Frozen, and of course listening to the array of songs that told the story of Elsa and Anna so perfectly. Like most children at the time, my favourite song was *'Let it Go'* sung by Idina Menzel, and I was devoted to learning every lyric to the song. In this essay, I would like to explore and investigate one line in the movie, sung in *'Let it Go'*. In the bridge of this iconic song, Elsa sings *"…spiraling in frozen fractals all around…"*. Studying mathematics in school and college, I have come across the term 'fractal' many times, but have never been informed of its meaning or prominence in mathematics. So, when listening to the song recently, and recognizing the terminology, I decided to delve into the world of fractals and ask myself 'Can fractals really be frozen?'.

What is a fractal?

Coined by a Polish-born French American mathematician, Benoit Mandelbrot, the term 'fractal' originates from the Latin *fractus* meaning 'broken' or 'fragmented'. Most people would refer to a fractal as a 'self similar' shape that is composed of smaller copies of itself, but that is only a generalised definition and not entirely true. A 2 dimensional shape is a shape that only has a length and width. Whereas a 3 dimensional shape is a shape that has length, width, and depth, and is the highest dimension we can visualise as humans. But what about a shape with a non-integer dimension? These shapes are classified as fractals and have a fractal dimension; not complex enough to be 3d but too complex to be 2d (or 2d and 1d respectively). The term fractal dimension refers to the relationship between the change in detail, and the scale of a shape, or a calculation of complexity. A remarkable feature this causes is that fractal shapes, in theory, can reach an infinite surface area or perimeter, while only occupying a finite and much smaller area or volume. And in fact, this concept is used in nature by trees, plants, and even the human body, which reflects Mandelbrot's ideology in inventing fractal geometry to model and represent shapes that had a 'roughness' to them, much like natural occurrences on earth.

The Von Koch Snowflake

The perfect fractal to begin this investigation is the Von Koch Snowflake or Von Koch Curve. First described by Niels Fabian Helge von Koch, the shape is constructed of equilateral triangles divided into segments of equal length and positioned in a repeating pattern like so.



(1)

Every shape defined as a fractal has its own unique dimension. But how do we find the dimension of this specific fractal? The general equation to find a fractal dimension is:

$$S^{D}=M$$

Where D is the fractal dimension, S is the scale by which each smaller repeat has been transformed by, and M is the scale by which the hypothetical mass has decreased, or how many smaller versions make up an identical, bigger shape. The curve is composed of 4 identical copies of the larger shape, and scaled 3 times smaller than the original. Therefore:

$$3^{D} = 4$$

$$\log_{(3)} 4 = D$$

$$D \approx 1.2618595$$

The fractional dimension of the Von Koch snowflake is approximately 1.2618595. This not only proves that the curve is a fractal, but also that it is self-similar, as we can easily apply this method to approximate its fractal dimension. However, a real snowflake would not be exactly self similar, as, in reality, an infinite surface area is impossible and nature often does not make formations identical to others.

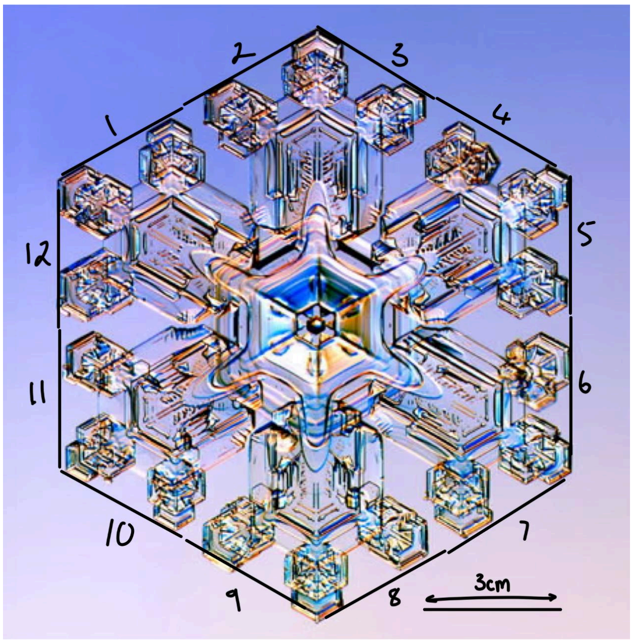
Fractals in nature

When Mandelbrot first thought of the concept of fractals, he not only thought of the theory behind the shapes but also their occurrences in everyday life. Defining a fractal is related closer to its dimension and roughness, rather than self similarity. This definition allows fractals to be a practical idea rather than theoretical, and help the way we understand nature and the world around us. Coming to a total of approximately 60,000 miles of blood vessels spreading across the human body, and the average human male having a height of 1.7m (approx. 0.00105633 miles), the question arises of how the human circulatory system fits into our bodies. By branching into smaller replicas of themselves, blood vessels can achieve a surface area that, in theory, approaches infinity while still occupying a finite and much smaller volume. This of course is intentional, as the circulatory system needs a large surface area to optimise the transfer of materials. And the circulatory system is not the only occurrence of fractals being used to maximise surface area in nature. Many trees and plants such as the Barnsley fern have adapted to use fractal formations to survive.

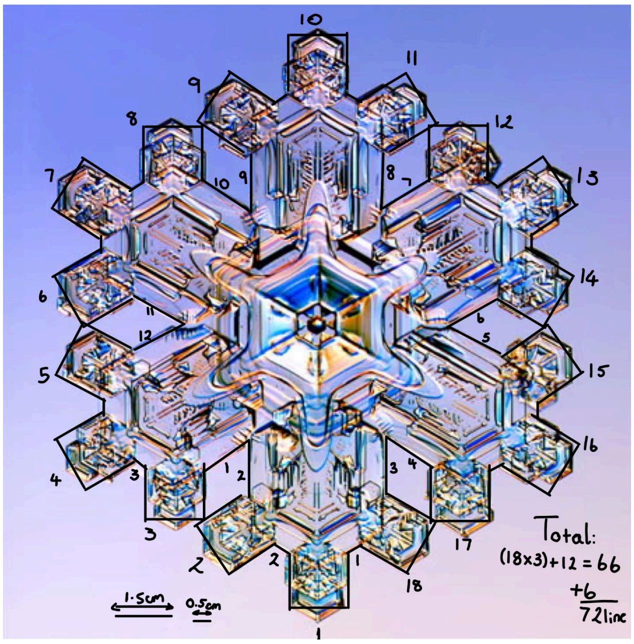
Snowflake Fractal Dimensions

One way to prove that naturally occurring snowflakes are fractals, is to find their fractal dimension. However, unlike infinitely self-similar fractals, we cannot use the simple equation used above, as the fractal must be precisely self-similar. As we know, in nature, no shape is a precise copy of another, and an infinite perimeter cannot be reached even on a molecular level. So, to find the fractal dimension we must think of the idea of dimension as the ratio of change in complexity to the scale at which the fractal is viewed. In this example, I have sourced a photograph from the internet of a snowflake under a microscope. I have then used measures of 3cm and 1.5cm to roughly outline the snowflake.

A

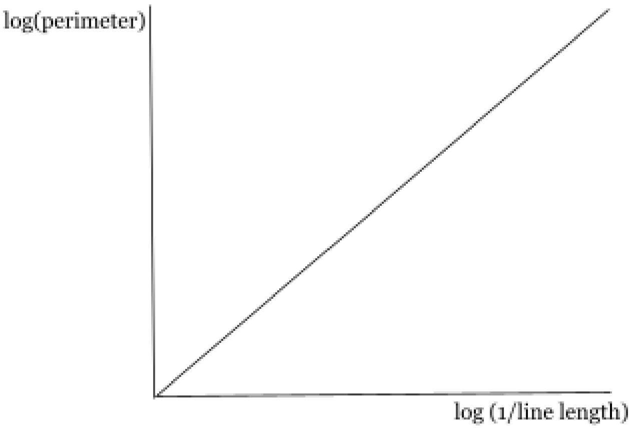


B



(2)

In image A, the perimeter is measured as 12, 3cm lines. In image B, the perimeter is measured as a total of 72, 1.5cm lines. Therefore the perimeter of A is approximately 36cm, whereas the perimeter of B is approximately 108cm. When we model this information as a graph of the log of the perimeter against the log of the reciprocal of the length of the line, we get a straight line. The fractal dimension can be calculated as the gradient of the graph.



[log(1.5) , log(108)]   [log(3) , log(36)]

$$m = \frac{log(108)-log(36)}{log(\frac{1}{1,5})-log(\frac{1}{3})}$$

$$m = 1.584962501$$

This implies that the fractal dimension of this snowflake is approximately 1.584962501. Therefore the snowflake is a fractal, even if it is not an exact self-similar shape and does not have a perimeter that tends to infinity. As I said at the beginning of this essay, the most known definition of a fractal is not the whole truth.

Limitations

As a mathematician, I must reflect on my findings and suggest any limitations to my model when finding the fractal dimension. Firstly, I have only compared 2 scales and perimeters, which, of course if I had much faster software and time, I could repeat the scaling of the lines multiple times to eliminate outliers and draw a more accurate graph. Secondly, I have found the fractal dimension of one snowflake. If I wanted to prove all snowflakes are fractals, I would have to test many other snowflakes using this process. Finally, there are many other methods to calculate fractal dimension which I could experiment with to verify my findings. Although there are limitations to my investigation, I believe that not only calculating a dimension that lies between 1 and 2 dimensions (which is appropriate for this shape), but also using a large difference in scale to make sure the gradient of my graph is as accurately calculated as possible, makes me confident in my findings and certainly answers the overall question of this essay.

Conclusion

My answer is yes, fractals can absolutely be frozen, and are actually found in nature quite frequently. Not only are snowflakes themselves fractals but ice formations and frozen rivers can also express themselves as fractals. In this essay, I have proven that my favourite childhood movie was more mathematically accurate than I thought, and that mathematics really is everywhere if we look closer. And closer. And closer. And closer. And closer. And closer. And closer. And closer

References

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