Catalytic Paradoxes: How Contradictions Spark Mathematical Evolution

By James Wright

Introduction

A paradox is a statement or argument that appears logically valid but leads to a self-contradictory or deeply counterintuitive conclusion. Paradoxes have been proposed throughout history in fields from philosophy to politics, but I intend to describe three mathematical paradoxes that focus on infinity and explain how they have developed our mathematical understanding.

1: Zeno's Paradoxes against Motion

Zeno of Elea (490–430 BC) is famous for three paradoxes which challenge the nature of motion: *The Dichotomy Paradox*, *The Paradox of Achilles and the Tortoise* and *The Arrow Paradox*. *The Dichotomy Paradox* involves trying to move from one side of a room to the other yet never succeeding, while *The Arrow Paradox* makes the argument that since an arrow is in a single position at each instant in time, it can never move. All three offer slightly different problems, however, I will describe *The Paradox of Achilles and the Tortoise* as an example.

Zeno states that Achilles is in a footrace with a tortoise and he gives the tortoise a 100m head start. By the time Achilles has run 100m to this point, the tortoise will also have moved a small distance forward, say 5m. By the time Achilles has run this 5m, the tortoise would have moved beyond this point again.

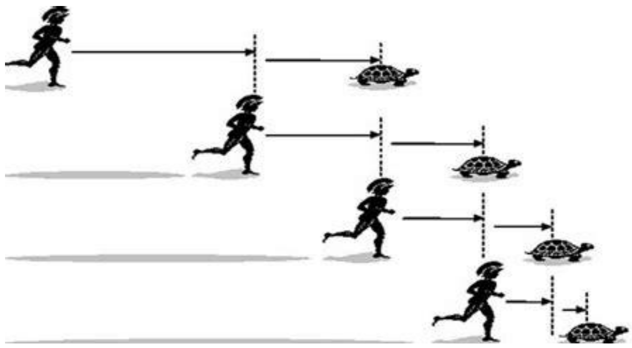

Figure 1: Achilles and the Tortoise

Repeating this process, Zeno makes the argument that in a race, the quickest runner can never overtake the slowest, since the quicker must first reach the point from where the slower started, so that the slower must always hold a lead.

These apparently trivial paradoxes have triggered responses and breakthroughs from some of the great mathematicians and physicists over the millennia. Intrinsic to Zeno's paradoxes against motion is the idea that a distance – such as that between Achilles and the tortoise – can decrease infinitely yet never reach zero. In the 17th century, Isaac Newton and Gottfried Leibniz challenged this mathematically by developing calculus.

$$\lim_{n\to\infty}\frac{1}{n}=\frac{1}{\infty}=0$$

They used limits to tackle the sum of infinite series and provide a finite result.

Furthermore, in 1899, Max Planck proposed the Planck length: a fundamental indivisible unit of length (approximately 1.6 x 10^{-35}m). This provided a physical solution to Zeno's paradoxes – the distance between Achilles and the tortoise cannot decrease infinitely as it will eventually reach the Planck length, so at some point Achilles will catch up the tortoise.

2: The Ross-Littlewood Paradox

This problem was first described by mathematician John Littlewood in 1953 and was expanded upon by Sheldon Ross in 1988. The paradox illustrates the conceptual difficulties with the notion of a supertask, in which an infinite number of tasks are completed sequentially.

The problem starts with an empty vase and an infinite supply of balls. An infinite number of steps are then performed, such that at each step 10 balls are added to the vase and 1 ball removed from it. To complete an infinite number of steps, it is assumed that the vase is empty at 11:59, and that steps are performed at 30 seconds before noon, 15 seconds before noon and so on with halving time intervals (step n is performed at 2^{-n} minutes before noon).

The question is then: *How many balls are in the vase at noon?*

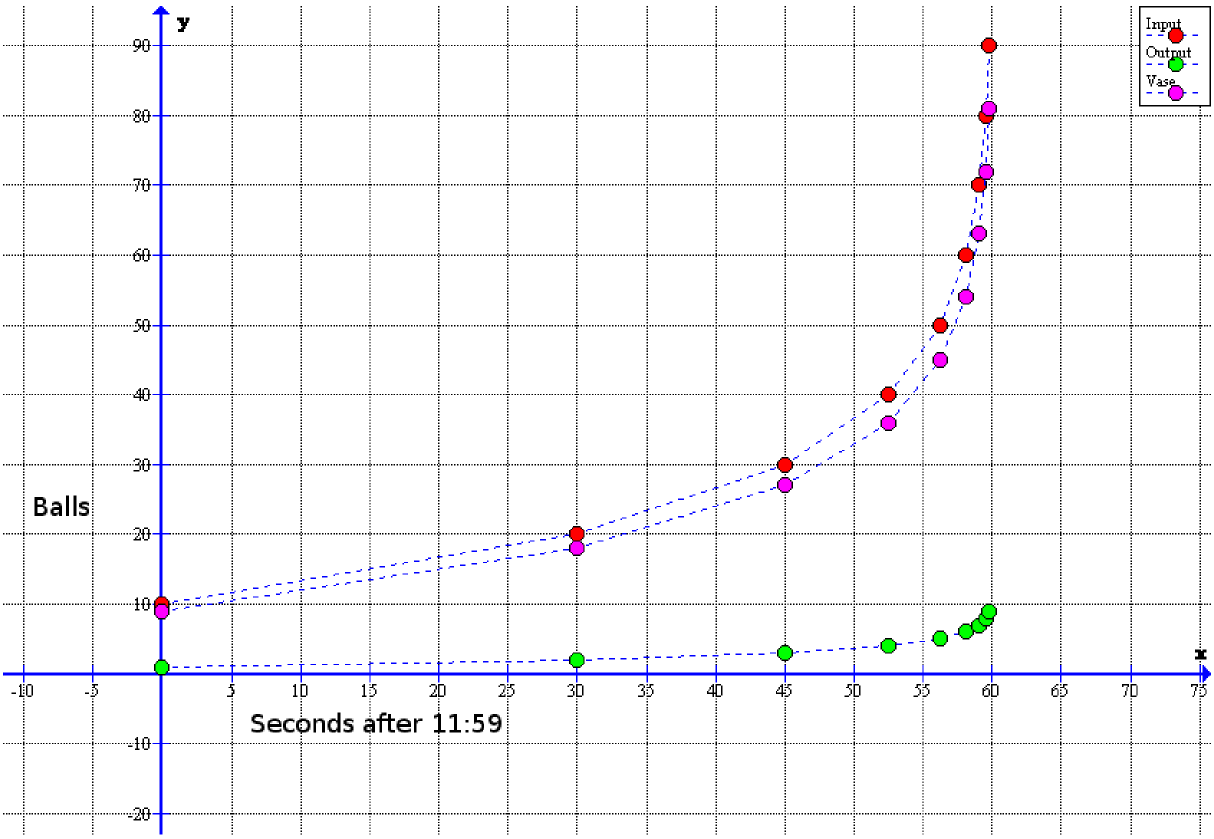

Figure 2: The times at which the first few balls are inserted into the vase

Intuitively you would assume that the vase contains an infinite number of balls at 12:00, because at each step the number of balls in the vase increases by 9 and there are an infinite number of steps.

However, it can be argued that the vase is empty. Imagine the balls from the infinite supply were numbered. At step 1 balls 1-10 are put in the vase and ball 1 is removed. Similarly, in step 2, balls 11-20 are put in the vase and ball 2 removed. By noon, every ball, n, that is inserted into the vase is eventually removed in a subsequent step – step n, and so the vase is empty.

Mathematically, both arguments make sense. If 9n balls are in the vase at step n, and $n \to \infty$, then there will be infinitely many balls in the vase at the noon. However, if 10n balls have been added and n balls removed by step n, then as $n \to \infty$, infinite balls have been added and infinite balls removed. These infinities have equal size as they are both the infinite sets of positive integers, and so subtracting one from the other gives 0.

Unfortunately, it turns out that these are not the only two possibilities; there is a method to end up with any number of balls in the vase at noon.

> Let x denote the desired final number of balls in the vase.
>
> Let n denote the number of the step currently taking place.
>
> For n = 1 to ∞:
>
> put the balls numbered from (10n - 9) to (10n) into the vase
>
> if n ≤ x then remove ball number 2n
>
> if n > x then remove ball number x + n

The first x odd balls are not removed, while all even balls or balls greater than or equal to $2x$ are. Therefore, exactly x balls remain in the vase.

As with Zeno's paradoxes, these problems arise from noon being unreachable. If infinitely many steps must take place before noon, then noon is a point in time that can never be reached. While the conditions are well-defined at every moment in time prior to noon, no conclusion can be made about the moment in time at noon.

This paradox has been addressed by many mathematicians since it was posed, such as Teun Koetsier and Jim Henle. It has led to a better defined and more rigorous approach towards infinite processes and has developed our understanding of the sizes of infinite sets.

3: Grandi's Series

In 1703, Italian mathematician Guido Grandi proposed the following infinite series.

$$\sum_{n=0}^{\infty}(-1)^n$$

This begins: 1 – 1 + 1 – 1 + 1 – 1 + ... It is a divergent series because the sequence of partial sums (1, 0, 1, 0, ...) does not converge to any number. Our task is to find the sum of Grandi's Series.

An intuitive method would be to pair up adjacent terms in the series which sum to 0.

$$(1-1)+(1-1)+(1-1)+(1-1)+\ldots=0+0+0+0+\ldots=0$$

However - as with *The Ross-Littlewood Paradox* - similar approaches can yield contrasting results. If the first term is not paired, the sum of the series is 1.

$$1+(-1+1)+(-1+1)+(-1+1)+\ldots=1+0+0+0+\ldots=1$$

But with a little more maths, a startling counter-intuitive solution arises. Using S to denote the sum of the series, the following process results in S = ½!

$$\begin{aligned} S &= 1-1+1-1+\ldots, \text{ so} \\ 1-S &= 1-(1-1+1-1+\ldots)=1-1+1-1+\ldots=S \\ 1-S &= S \\ 1 &= 2S, \end{aligned}$$

And so S = ½.

The same result can be found if we apply the equation for the sum to infinity that we would use for converging series.

$$\lim_{N\to\infty}\sum_{n=0}^{N} r^n=\sum_{n=0}^{\infty} r^n=\frac{1}{1-r}.$$

However, this is only valid when r is strictly within the interval (-1,1), so I will use (-1+ε) for r.

$$\sum_{n=0}^{\infty}(-1+\varepsilon)^n=\frac{1}{1-(-1+\varepsilon)}=\frac{1}{2-\varepsilon},$$

Then as ε approaches 0:

$$\lim_{\varepsilon \to 0} \lim_{N \to \infty} \sum_{n=0}^{N} (-1 + \varepsilon)^n = \frac{1}{2}.$$

Grandi himself developed yet another approach, by using the following binomial expansion.

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \ldots$$

He then substituted in $x = 1$ to give the sum of the series as ½.

But this paradox was not merely to be an interesting peculiarity. *Grandi's Series* inspired the 19th century Italian mathematician Ernesto Cesaro to investigate ways to assign a value for the sum of divergent series. This became known as Cesaro Summation.

The Cesaro sum of a series $a_1 + a_2 + a_3 + \ldots$ is the limit as n approaches ∞, of the arithmetic mean of the first n partial sums $s_n = a_1 + \ldots + a_n$:

$$\lim_{n \to \infty} \frac{s_1 + s_2 + \cdots + s_n}{n}$$

In the case of *Grandi's Series*, the partial sums are (1, 0, 1, 0, 1, 0, ...). The sequence of arithmetic means is:

$$\left\{ \frac{1}{1}, \frac{1}{2}, \frac{2}{3}, \frac{2}{4}, \frac{3}{5}, \frac{3}{6}, \ldots \right\}$$

which has the limit ½. Thus, the Cesaro sum of *Grandi's Series* is ½.

Conclusion

Many people view paradoxes merely as interesting puzzles for philosophers to grapple with, or just as fun riddles to tell one's friends. However, in the three examples I have investigated, I hope to have demonstrated that paradoxes play a crucial role in the development of mathematics. Such problems have intrigued mathematicians for millennia and have forced them to make countless breakthroughs in order to put to rest some rather counterintuitive and sometimes outright disturbing claims.

Whether it is Zeno's continual questioning fuelling calculus and quantum theory, Littlewood's strange ideas challenging the nature of infinity or Grandi's deceptive sums probing mathematicians' understanding of diverging series, paradoxes have moved maths a long way forward.

Although Zeno might tell you that we have moved nowhere at all...

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