A FURTHER MATHEMATICS ESSAY

Circular Motion

From uniform orbits to the edge of a rollercoaster loop

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There is something almost deceptively simple about an object moving in a circle. A ball on a string, a car going round a bend, the Moon — these all feel like fairly mundane examples. But underneath each of them is a physical idea that's actually quite easy to miss: that you can be accelerating even while your speed stays constant. This trips a lot of people up at first, and honestly fair enough, because it contradicts the intuition that acceleration means speeding up or slowing down. It doesn't. Circular motion is probably where this becomes clearest.

What follows builds up the main results from something close to scratch. It's aimed at further maths level — so the reasoning should be reasonably careful, there'll be some calculus-adjacent arguments, but nothing that requires much beyond A-level mechanics and a willingness to think about vectors properly.

1. The Language of Circular Motion

Before equations, we need the right language. A particle moving around a circle of radius r has its position described at any moment by an angle θ, measured in radians from some reference direction — it doesn't really matter which one. The rate at which this angle changes is the *angular velocity*, ω (omega), in radians per second. For uniform circular motion — the easy case we start with — ω is constant throughout.

The linear speed v of the particle relates to ω and r through what is probably the single most-used formula in this topic:

LINEAR SPEED $v = r\omega$

The argument is pretty clean: in one full revolution the particle covers $2\pi r$, sweeping 2π radians, taking time T (the period). So $v = 2\pi r/T$. But since $\omega = 2\pi/T$, substituting straight in gives $v = r\omega$. We also write frequency as $f = 1/T$, so $\omega = 2\pi f$ — worth knowing for when problems hand you frequency instead of period.

2. Centripetal Acceleration

Here’s the bit that actually matters. Even if a particle goes round a circle at perfectly constant speed, its velocity is still changing — because velocity is a vector and the direction keeps shifting. A changing velocity means acceleration, full stop, regardless of whether the magnitude is constant. This confused people for a long time, and it’s worth sitting with rather than just accepting.

To get the acceleration properly, look at velocity vectors at two nearby moments, separated by a small time δt. Both have magnitude v, but they point in slightly different directions — separated by an angle $\delta\theta = \omega\,\delta t$. The change in velocity δv points roughly toward the centre. For small angles its magnitude is approximately:

CHANGE IN VELOCITY (SMALL ANGLE) $|\delta v| \approx v \cdot \delta\theta = v \cdot \omega\,\delta t$

Dividing by δt and letting $\delta t \to 0$, you get the magnitude of the acceleration as:

CENTRIPETAL ACCELERATION $a = v\omega = v^2/r = \omega^2 r$

All three forms are equivalent — each is useful in different situations, so memorise all of them. The direction is always toward the centre; *centripetal* literally means centre-seeking in Latin. And importantly, this isn’t some exotic new kind of acceleration — it’s just the ordinary acceleration vector, which happens to point inward when the path is circular.

3. Centripetal Force

Newton’s second law says $F = ma$, so if there’s acceleration toward the centre there must be a net force toward the centre. That force is the centripetal force:

CENTRIPETAL FORCE $F = mv^2/r = m\omega^2 r$

Here’s something that causes endless confusion on exams: centripetal force is not an actual new force. It’s just the label for whatever real force happens to be acting inward in a given situation. In different scenarios, different physical things provide it:

A ball on a string — tension. A car rounding a flat bend — friction from the road acting sideways. The Moon orbiting Earth — gravity. The maths is the same each time, but you have to correctly identify which real force is doing the job. Getting this wrong is probably the most common source of errors in circular motion questions.

On "centrifugal force": People often talk about the centrifugal force that pushes you outward on a roundabout or in a spinning car. In an inertial frame this force doesn't exist — what you're feeling is just inertia, your body trying to continue in a straight line while the car curves away underneath you. It's a real sensation but not a real force, at least not in our standard Newtonian framework.

4. A Worked Example: The Conical Pendulum

A good example to work through is the *conical pendulum*: a bob of mass m on a string of length L, swinging in a horizontal circle with the string at angle α to the vertical. The bob moves in a circle of radius $r = L \sin \alpha$.

Two forces act — weight mg downward and tension T along the string. There's no vertical acceleration, so vertically:

VERTICAL EQUILIBRIUM $T \cos \alpha = mg$

Horizontally, the inward component of tension is what provides the centripetal force:

HORIZONTAL (CENTRIPETAL) $T \sin \alpha = m\omega^2 r = m\omega^2 L \sin \alpha$

Dividing the second equation by the first (and cancelling sin α, assuming it's nonzero):

ANGULAR VELOCITY OF THE CONICAL PENDULUM $\omega^2 = g / (L \cos \alpha)$

Which is a fairly neat result — the mass cancelled out entirely, so ω depends only on the geometry and g. There's also a nice limiting case: as $\alpha \to 0$, $\omega \to \sqrt{g/L}$, which is just the angular frequency of a small-angle simple pendulum. Not a coincidence, more like the two problems being secretly the same thing in different clothing.

5. Motion in a Vertical Circle

Things get more interesting — and honestly more fun — when the circle is vertical. Now gravity isn't perpendicular to the motion the whole way round, so the speed varies and you can't just apply $F = mv^2/r$ once and be done. This is the physics behind rollercoaster loops, the classic "bucket of water over your head" demonstration, and the Wall of Death.

Take a particle of mass m on a string of length r, swinging in a vertical circle. Let v_0 be the speed at the bottom. Using energy conservation (zero potential energy at the bottom), at angle θ from the bottom the speed satisfies:

ENERGY CONSERVATION (VERTICAL CIRCLE) $\tfrac{1}{2}mv^2 = \tfrac{1}{2}mv_0^2 - mgr(1 - \cos\theta)$

At any point on the circle, Newton's second law along the radius (taking inward as positive) gives:

EQUATION OF MOTION (GENERAL POINT) $T - mg\cos\varphi = mv^2/r$

where φ is the angle the string makes with the vertical — so at the top $\cos\varphi = -1$, meaning gravity is now pulling inward, which is why that's the critical point.

The key question is what minimum speed is needed at the top for the string to stay taut. Taut means $T \geq 0$, and the boundary case is $T = 0$:

MINIMUM SPEED AT THE TOP (T = 0) $mg = mv^2_{\text{top}} / r \implies v^2_{\text{top}} = gr$

Plugging back into the energy equation between bottom and top (height difference of 2r):

MINIMUM LAUNCH SPEED AT THE BOTTOM $v^2_0 = v^2_{\text{top}} + 4gr = 5gr \implies v_0 = \sqrt{(5gr)}$

So $\sqrt{(5gr)}$ at the bottom is the threshold. Drop below it and the string goes slack somewhere before the top, at which point the particle becomes a projectile and the situation gets complicated fast. Worth noting: for a particle on the inside of a smooth track rather than a string, the same condition holds, just with normal reaction replacing tension.

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6. Banked Curves and Real-World Applications

A nice application that comes up in exams is the banked road curve — a bend where the road is tilted inward at angle β. The idea is that at the right speed, the horizontal component of the normal reaction alone handles the centripetal force, and you don't need any friction at all. Resolving vertically gives $N \cos \beta = mg$, horizontally gives $N \sin \beta = mv^2/r$, and dividing:

IDEAL BANKING ANGLE $\tan \beta = v^2/(rg)$

This is genuinely used in road design — motorway junctions and racing circuits have banking angles calculated from typical speeds. If you go faster than the design speed friction starts doing work, slower and it works the other way. The formula is clean enough that it's worth just knowing off the top of your head.

Satellites are another natural application — a circular orbit of radius R from Earth's centre has gravity providing the centripetal force. Setting $GMm/R^2 = mv^2/R$ gives orbital speed $v = \sqrt{(GM/R)}$. Geostationary satellites need a 24-hour period which pins down their orbital radius to around 42,000 km. Again the same equation, different physical context.

Conclusion

Circular motion isn't really a topic in its own right so much as a particularly clean application of Newton's laws to curved paths. The central result — that constant speed still gives centripetal acceleration v^2/r inward — follows directly from vectors, and everything else builds on that. Whether it's a pendulum, a rollercoaster, or a satellite, the same two or three equations cover it.

The main thing to get right, practically speaking, is identifying the real force providing centripetal acceleration in a given problem and resolving carefully — not just writing $F = mv^2/r$ without thinking about what F actually is. Do that consistently and most circular motion questions become fairly routine, even if they don't always feel like it at first.