What goes up must come down… possibly

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Introduction

The Collatz Conjecture, often called the 3n + 1 problem, is a deceptively simple mathematical process that has challenged mathematicians for over 80 years. Despite its straightforward rules, no one has yet proven that it always works, making it one of the most famous unsolved problems in mathematics.

The process begins by choosing any positive integer, n. If n is even, divide by 2. If n is odd, multiply by 3 and then add 1. Repeat the process using the new number. The conjecture states that for any positive integer chosen, the sequence will eventually reach 1. At 1, the process enters a repeating cycle of $4 \to 2 \to 1 \to 4 \to 2 \to 1$ and continues indefinitely.

At first glance, this claim appears simple enough to verify. Using computers and brute-force calculations, mathematicians have checked that the conjecture holds true for all numbers up to 2^{68}. Although this number is extraordinarily large, it is still insignificant compared with the infinite set of positive integers. Therefore, computational evidence alone is insufficient in proving the conjecture is always true.

Why this “simple” problem refuses to be solved

One of the greatest challenges in proving the Collatz Conjecture is that mathematicians cannot construct a general proof that applies to all numbers. Instead, mathematicians often attempt to disprove the conjecture by searching for a contradiction - specifically, a number that won’t eventually reach 1.

A contradiction could occur in two ways. First, a sequence might increase without bound (diverge), growing infinitely large. Second, the sequence might fall into a repeating cycle like the familiar $4 \to 2 \to 1$ loop.

Interestingly, when the 3n+1 process is applied to negative integers, several alternative repeating cycles have been discovered. For example:

$$-5 \to -14 \to -7 \to -20 \to -10 \to -5$$

The existence of cycles for negative numbers suggests cycles could theoretically exist for positive integers. However, since the conjecture has been verified for numbers up to 2^{68}, any undiscovered cycle would involve numbers much larger than those already tested. Estimates suggest that such a loop would contain billions of numbers, rendering it extremely difficult to detect through computation alone causing mathematicians such as Paul Erdos to say, “Mathematics is not yet ripe enough for such problems”.

A Bias Towards One:

Although no formal proof exists for the Conjecture, there are heuristic arguments which suggest that numbers decrease on average. This can be seen below:

If n is even, the multiplier is ½.

If n is odd, the multiplier is 3

This process would appear to lead to an average increase in n over time. This is incorrect however, as the chance of the next number in the sequence being odd or even is not equal. As 3n + 1 becomes even when n is odd, the next step divides the result by 2. Therefore, the multiplier for odd numbers is $\frac{3}{2}$ (the +1 becomes insignificant for large values of n).

To estimate the long-term behaviour, we assume that numbers are equally likely to be odd and even after the above process. We can then calculate the geometric mean:

$$\sqrt{0.5 \times 1.5} = 0.866$$

which gives a multiplier of 0.866. This does not prove the Conjecture because it is only statistical. There is also a possibility that another loop exists. Statistics prove this generally works, but this conjecture is not completely random – it follows a strict set of rules that create patterns.

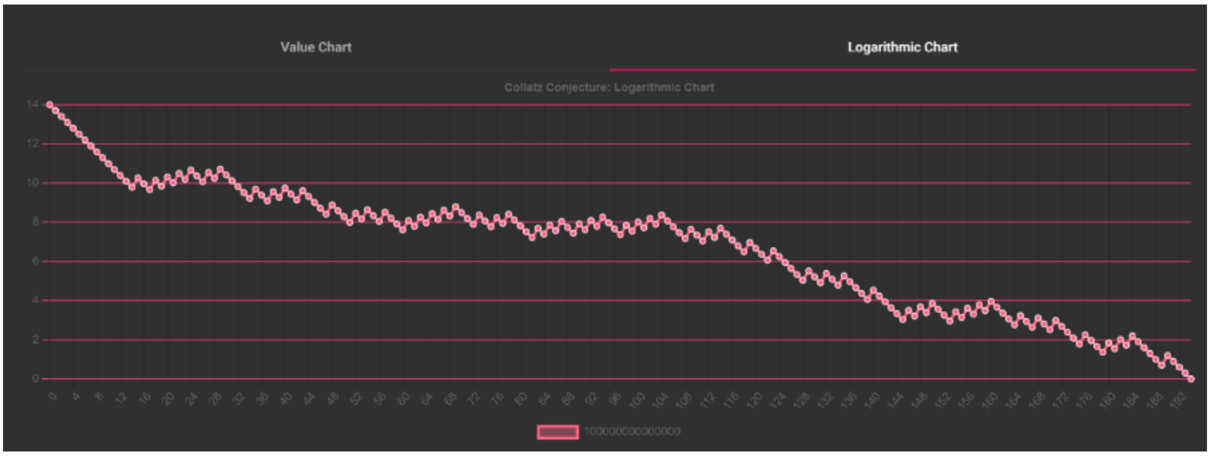
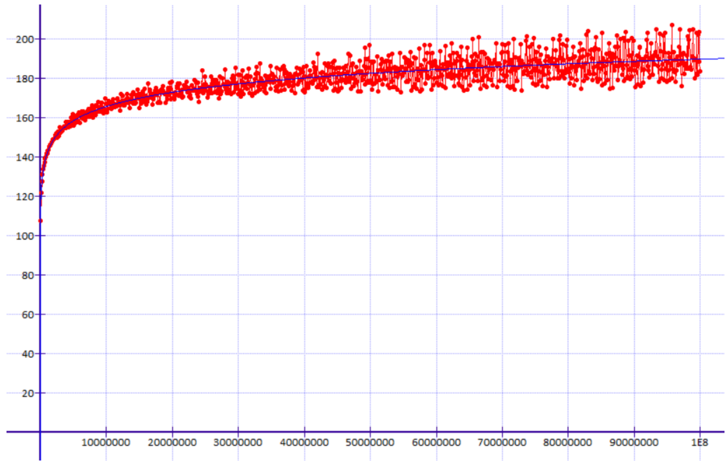


Figure 1: Shows the Collatz Conjecture's decreasing random walk in log form (1)



$S = 10.45 \ln n - 3$

Figure 2: Average number of steps required for numbers in the Collatz Conjecture to reach 1. The values of n were grouped in intervals of 100,000 and the average number of steps were calculated for each interval, up to 100,000,000.

Although the average number of steps follows a clear logarithmic relationship, there is still large variance from the line, even after averaging values within intervals of 100,000.

The equation produced can estimate steps required to reach 1 for a value of n, however, it is not perfectly accurate as individual values in the 3n+1 process vary significantly.

Terence Tao's breakthrough:

In 2019, Tao produced a paper on the Conjecture. His paper, "Almost all Collatz orbits attain almost bounded values" (2) approached the Conjecture from a different angle. He proved that for almost all numbers, they eventually decrease to smaller values. He did this by using probability, logarithmic methods, and statistics rather than numbers individually.

While this sounds particularly vague, Tao's work showed that whilst it is possible some sequences diverge, if these sequences exist, they must be extremely rare. It also suggests that numbers will almost certainly end up at or below the starting value. His work is not specific to repeating loops in the Collatz Conjecture.

This idea comes from logarithmic density, which involves measuring how large a set of numbers is compared to the entire number system when weighted by their size. Tao showed that the set of numbers that fail to decrease sufficiently has logarithmic density zero. As they have this property, it means they are so scarce that they will barely appear when numbers are sampled across large ranges. Put simply, it shows that divergence (numbers growing without bounds) is statistically unlikely on a massive scale.

Most work on the Conjecture before Tao focused on numerical behaviour such as stopping time (number of steps to get to 1) and cycles attempting to prove patterns that fit for all numbers. Tao instead treated it like a random system that doesn't follow a strict deterministic path. He treated the problem like a random walk and then used logarithmic scaling (multiplying by 3 becomes adding $\log 3$, dividing by 2 becomes subtracting $\log 2$). Each step in the Conjecture becomes a walk forward of $\log 3$ or a walk backward of $\log 2$. We know that this walk is biased towards the even side due to the above proof and therefore downward movements dominate over time.

Using this, Tao showed that most Collatz sequences eventually fall below certain thresholds and that the Collatz sequence eventually becomes smaller than

$$n^{o(1)}$$

This means that the sequence eventually falls below any equation (including incredibly slowly growing equations).

Importantly, this does not prove that all sequences reach 1, instead that they do not usually increase indefinitely, weakening the possibility of infinite growth.

Eliminating large classes of possibilities in mathematical research is a major form of progress. Tao's paper has caused many researchers to be interested in statistical behaviour, average-case analysis and probabilistic modelling associated with the 3n + 1 problem.

John Conway's findings:

Conway studied generalised processes of the Collatz Conjecture. As the 3n + 1 problem became well known, similar functions known as Collatz functions were investigated. Conway showed that there are Collatz functions that exist which are fundamentally undecidable (discussed below).

Instead of using the classic Collatz Conjecture rules:

Odd -> 3n+1 Even -> /2

Conway considered systems similar but with different rules such as:

Odd-> 5n-1 Even -> /2

Or instead of odd or even, looking at remainders when dividing n by a specific number. Using these, Conway proved that some Collatz-type functions are undecidable, which means it is impossible to prove it true or false. The idea of undecidability originally came from Alan Turing on the Halting problem.

The heart of what Conway was trying to discover was that generalised Collatz functions can simulate computation. This is because numbers evolving through rules act like instructions. This is known as computational universality. A system is computationally universal if it can simulate any computer program or algorithm which many Collatz-like functions can do.

Conway's work allows us to conclude if a Collatz-like function can simulate a computer, It can simulate problems that are already known to be undecidable, therefore, some Collatz like problems must be undecidable.

Conway did this by encoding information into numbers, then altered this information in steps, mimicking computational operations.

$$n \to 3n + 1 \to \frac{3n + 1}{2}$$

$$Instruction \to Instruction \to Instruction$$

Essentially, Conway showed functions/systems could become programs.

Conway's work showed mathematicians that Collatz type problems can be as powerful as computers, instead of being arithmetic puzzles.

Conway was not solely studying Collatz-type functions, instead his work connected to several major mathematical fields and Collatz functions became an essential part of his research.

Conway showed that the Collatz functions are connected to the Halting problem; will a computer program eventually stop? Which is proven to be undecidable.

Since Collatz-like systems can simulate programs, they inherit this undecidability.

Collatz-like Sequences:

As discussed above when writing about Conway's work, there are many Collatz-like sequences.
An example of one, is 5n+1:

Odd -> 5n+1 Even -> /2

Interestingly, unlike 3n+1, this problem often diverges. 5n+1 behaves differently to 3n+1 in this situation because the multiplier of 5n+1 is larger than 1, therefore on average after many steps, n increases instead of decreasing with 3n+1.

You can create many Collatz-like sequences and test if they are divergent (go towards infinity), convergent (go towards 0/1 (depending on the numbers used)), or cyclic (form loops).

A visualisable tool can be used for Collatz-like problems. It shows which numbers are divergent, convergent or neither. (3)

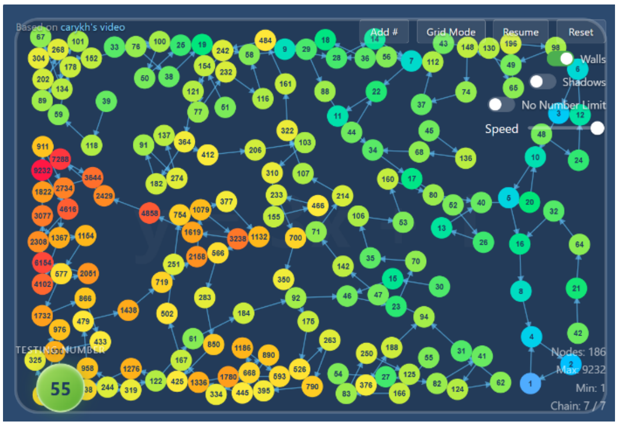


Figure 3: The program chains together numbers. The arrow is a step. It tries to show every path which links to a number (e.g. 16 will be connected with 32 and 5 as those are the 2 numbers which can make 16 in the Collatz Conjecture.

When using grid mode, you can compare Collatz-like sequences. The top row is 1n+c, with +c going from 1-5 left to right. The second row is 2n+c etc, finishing with 5n+c on the bottom. Integer overflow means that a number has gone above 2.147 billion (integer overflow on Java). If a number has done this, it is very likely to have diverged. Interestingly, a criss-cross pattern is produced which is explained below.

If n is odd and is multiplied by an even number and then an odd number is added to it, it will stay odd and the same process will repeat again. This creates an exponentially increasing sequence. If n is even, it will be halved until it is an odd number and then the exponentially increasing sequence will begin. This means it will diverge.

If n is odd and is multiplied by an odd number and then an even number is added to it, it will stay odd and also diverge.

The sequences above diverge for every value of n, because once they are odd, they stay odd for every future step.

1n+2 and 1n+4 are the same situation, they just haven’t reached integer overflow yet as they take longer to increase, however, they will also diverge.

We have already shown that for some functions, such as 5n+1, its multiplier is larger than 1 and therefore often will diverge. This is therefore the same for 5n+3 and 5n+5, however, there may be loops in these sequences and therefore one cannot say these sequences are divergent for all values of n.

Other sequences such as 1n+1, 1n+3, 1n+5 will be convergent, as when n is odd it is multiplied by 1.

1n+1 loops in a sequence of $1 \to 2 \to 1$

1n+3 loops in a sequence of $1 \rightarrow 4 \rightarrow 2 \rightarrow 1$

1n+5 loops in a sequence of $1 \rightarrow 6 \rightarrow 3 \rightarrow 8 \rightarrow 4 \rightarrow 2 \rightarrow 1$

Overall, the Collatz Conjecture demonstrates how a problem with simple rules can lead to such fascinating behaviour. Despite being tested up to 2^{68} and studied by leading mathematicians, a complete proof remains completely out of reach. Researchers such as Terence Tao show that progress is still possible, even without solving the problem. Meanwhile, the investigations of Conway into generalised versions of the problem reveal that similar systems are undecidable. I managed to show method in the madness of Collatz-like functions. Why some sequences diverge, some converge and with others it becomes difficult to tell (such as 3n+1).

The Collatz Conjecture is a useful tool for education, introducing students to unsolved problems and demonstrating experimental mathematics. This has made Collatz sequences a popular introduction to research style mathematics.

So, if you're looking to study a niche topic of mathematics that you'll most likely never be able to solve, the Collatz Conjecture is for you!

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