Is there a connection between Fibonacci numbers and the stable cycles of the logistics map?

Mathematics always contains astonishing undiscovered links. Some of which take more time and effort to uncover, but nonetheless, between concepts which lay on opposite ends of the simplicity and complexity spectrum, but it roots in both with the simple equations eventually leading into chaos despite the initial facade of order. The logistics map and the Fibonacci sequence are two well known examples of this. The logistics map is a model based on an equation, x(n+1) = r(x(n)) (1−x(n)). The output of the function depends on the value of the parameter, which is r, the instigator of chaos, which controls the system in which it portrays chaotic, unpredictable behaviour from a simple equation. X represents the population at a given time, n, which ranges from 0 to 1. When r is less than 3.57 the behaviour is predictable, allowing you to know where the population is at; however after 3.57 the behaviour of the population becomes chaotic, which can be shown on the logistics map bifurcation diagram. [1] The Fibonacci sequence is a common, known sequence where the next value is the sum of the two previous values, making outcomes predictable and has a clear, structured nature to it, something the logistics map highly lacks. The first few terms of this sequence are 0,1,1,2,3,5,8,13,21 and so forth to infinity. Both of these mathematical concepts have different natures to them as one leads to chaos while the other has a pattern and structure, making it difficult to discover a relationship between these two. When both are displayed on graphs it's hard to figure out how these concepts could possibly link, given there is no direct numerical relationship between the Fibonacci numbers and the stable cycles of the logistics map; however one fundamental point is that both of these models come about from an iterative process, where the next value depends on the previous value, allowing a link to be made between these two ideas.

Within the logistics map, as the value of r increases the system loses stability, where the system stops converging towards a single value and gradually starts to oscillate between a fixed number of values of 0 and 1, the time, initially between two values, then later at four values, then at 8 values, 16, 32, etc[2]. This process is known as the period doubling until r is more or equal to 3.57, where the structure can be characterised as fractals. From this point after the system becomes chaotic as even the slightest change in parameters drastically changes the outcomes. As stated in the Chaos: Making a New Science, there is a ratio between the successive intervals of the value of r, the intervals between the successive bifurcations shrink at a consistent rate as it approaches a constant, the Feigenbaum constant which is approximately 4.669, indicating that the chaos is actually not purely random, as Gleick emphasises the discovery disregarding the assumption that the chaotic system was inherently disordered, but instead follows mathematical laws, which seems ironic when looking at the graph but "simple systems give rise to complex behaviour"[3].

The Fibonacci sequence is shown by the equation F(n+1)= F(2) + F(n-1)[4]. This has elements of consistency and sets predictable outcomes with a clear pattern. Unlike the logistics map, a notable property of this sequence is that the ratio of the consecutive terms converges towards a constant value,

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[1] May, Rober. *Simple mathematical models with very complicated dynamics*, vol. 261, no. June 10 1976, 1976, p. 1. *Islands of order*,

[2] May, Rober. *Simple mathematical models with very complicated dynamics*, vol. 261, no. June 10 1976, 1976, p. 3. *Islands of order*,

[3] Gleick, J. *Chaos: Making a New Science. New York: Viking.* 1987, p. 304

[4] Chasnov, Jeffrey. *Fibonacci numbers and the Golden ratio*. 2016. *The Hong Kong University of Science and Technology Department of Mathematics*.

which is the golden ratio. In the Fabulous Fibonacci numbers book, the converging is further explored where Posamentier highlights how the sequence is used for various reasons or appears in nature, from architecture and paintings such as the Mona Lisa to even sunflowers and hurricanes, making it evident that this ratio is not just a mathematical wonder, but signifies its value which can instead explain the reasoning for why the system behaves that particular, chaotic way. This idea is also further explained in The Golden Ratio: The Story of Phi. This portrays the golden ration as a recurring proportion in geometry, art, nature and the body itself, instead of as a limit as Livio is carefully able to distinguish the differences in the significance and the exacerbated claims of mathematics, indicating the value he holds towards the ratio is highly invaluable showing a coherence in the unrelated systems.

One of the obvious and crucial connections between the logistics map and the Fibonacci sequences is that they both rely on iteration, where each term is generated from the previous values meaning the system comes about from a fixed step by step rule but the process through which you generate outcomes differs significantly. In the Fibonacci sequence, the cyclic procedure shows a stable and predictable pattern, where each term reinforces the structure of the sequence as a whole, as the next term is the sum of the two previous terms, ensuring the previous terms affects the terms to come. In the logistics map the same principles and concept applies but produces behaviour where the output becomes highly contagious and sensitive to changes in parameters, subsequently resulting in chaos. This unforeseen outcome suggests that the recursion does not only define the exhibiting nature of the system, but is rather a form of a rule being applied that determines the long term behaviour of the system. Gleick's take on the chaos theory, "chaos was the end of the reductionist program in science"[5] reinforces the point, showing how even a minor change in mathematical rules largely differs the outcomes. Even a 0.00001 difference.

Another comparative point is based on scaling. The logistics map executes a form of self similarity, especially in its bifurcation diagram, where the patterns repeat of the periods but just getting into smaller scales, which links closely to the Feigenbaum constant, which essentially explains how the spacing between the bifurcation decreases in a regular and predictable way. This structure indicates an alternative way of looking at it, a hidden order behind the chaotic initial appearance. The Fibonacci sequence never contains any chaotic attitudes and elements, but it also similarly reflects the scaling element. As the sequence progresses, the ratio of the terms becomes stable, which governs the proportions of terms within the sequence. Livio's findings of the golden ratio emphasised that this form of scaling is not arbitrary, but instead emerges naturally from the recursive formula of the sequence.

However, aside from the significant similarities, when looking at the long term behaviour, there is still a stark difference as the Fibonacci sequence is completely set and deterministic on its planned route and resides in a transparent atmosphere, once the two terms are acknowledged, every subsequent value will inevitably follow from the iterative process, allowing no room for ambiguity or divergence, ensuring that the sequence maintains the stable progress and the predictable manner. Whilst the logistic map is also determined and strict within its theory and formulaic rules, the behaviour of it becomes almost impossible to predict due to the extreme sensitivity to its initial conditions. As the parameter r increases beyond certain ranges, even infinitely small differences such as 1 x 10^(-100) in the starting value can lead to completely different outcomes, showcasing the sensitive dependent nature on the initial conditions which is vital to the chaos theory, as discussed in the Chaos: Making a New Science, which reinforces the deterministic quality of the system being able to produce random

[5] Gleick, J. *Chaos: Making a New Science. New York: Viking.* 1987, p. 308

behaviour, identifying the moral that determinism is not causal to predictability, so despite having fixed rules, only the Fibonacci sequence can realistically be anticipated.

A further point of divergence is that in the Fibonacci, the golden ratio emerges as a limiting value that showcases the relationship between the successive terms, indicating that the convergence is smooth and stable and when it is reached the ratio remains consistent. In The Golden Ratio: The Story of Phi the golden ratio has been widely studied through mathematical properties and has been "described as the most irrational of all numbers"[6] where it is also appeared in geometry and nature, so Livio takes a careful approach to with caution when handling the claims, acknowledging that value of the golden ratio and its uniqueness and the significance it has in maths being undeniable, even when "many claims about the golden ratio are exaggerated"[7] indicating the prevalence of the golden ratio can often be overstated. In the context of the logistics map, the Feignebaum constant does not indicate a stable relationship between the values and instead shows the rate at which the system transforms and shifts into chaos, being able to capture how quickly the intervals between bifurcations decrease as the more of the period doublings occur, reflecting the approach to instability, making it difficult to directly compare the two constants despite both of them being able to apply universally when they are associated with their respective mathematical concepts.

The way these two systems are typically approached at and applied at differ as the Fibonacci sequence is frequently associated with efficiency and optimality, particularly in biological systems such as the inheritance tree of the human X chromosome, the number of spirals on pinecones, pineapples and certain flowers is always a Fibonacci number or the lengths of the bones in the human finger being proportionate to the Fibonacci numbers[8]. As noted in The Fabulous Fibonacci Numbers, they come about in contexts such as branching patterns and leaf patterns, where they appear to provide structurally efficient solutions. Despite some claims of their presence in nature being exaggerated, the evidence is too loud and vibrant to be ignored and disregarded, which makes it inevitably clear of the recursive growth patterns leading to arrangements which minimise overlapping in nature and biology and maximise exposure in areas such as architecture and geometry such as combinatorics. As expected, the logistics map fails to demonstrate these same attributes, but instead displays its own form of being unstable especially in systems such as the population dynamics where it doesn't account for the environmental factors. His interpretation on the theory indicates the systems customary assumptions being opposed based on predictability claiming that even simple equations can produce irrational behaviour that becomes almost impossible to forecast. The difference in the interpretations of these systems makes them more incomparable as they assimilate distinct roles, where one as representative of structured growth and the other as a reflection of complexity and unpredictability.

Instead of treating the stable cycles of the logistics map and the fibonacci sequence as directly compatible, it would make sense to look it at from another angle where they both emerge from the same concept as they are both built on iteration but there is yet to be a demonstration where the repeating is a simple rule which leads to a single type of outcome. In one case, the process is stabilised where the pattern becomes more clear and evident but in the other it initially follows the period doubling pattern, progressing through powers of two and then drifts into a behaviour that is that complete opposite of long term stability. So whilst there was no direct connection between the

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[6] Livio, M. *The Golden Ratio: The Story of Phi*. 2002, p. 73. *London: Headline*
[7] Livio, M. *The Golden Ratio: The Story of Phi*. 2002, p. 5. *London: Headline*
[8] Kimberly, Rivera. *The Fibonacci Sequence.* 2017, p. 6-8. *Demonstrating the Magic of Math*

Fibonacci sequence and the stable cycles of the logistics map, looking at them side by side makes it clear that they both share the capacity to be able to generate complexity from simplicity, even when solely based on the same kind of repetition. Seen this way, they sit on opposite sides of the spectrum yet both demonstrate how much can emanate from very little.