Ever been sat in maths class typing an equation into the solver bit of your calculator, praying your equation's correct and you don't get scary complex numbers? Come on, we've all done it. But it turns out complex numbers aren't all that scary – I actually think they're quite nice. (It's true, I promise! Stay with me here.)

First of all, what is a complex number? Well, you know how square numbers are always positive? Turns out that's only true if the square root is a *real* number, like 2, or -1, or 3.14159265. Any number on the traditional number line, positive or negative, integer or decimal, rational or irrational, is real.

An imaginary number is the square root of a negative number. We use i to represent $\sqrt{-1}$. Imaginary numbers exist on a different number line (it's perpendicular to the real number line, but more on that later). And when we combine real and imaginary numbers, we get a lovely complex number, like 2 + i, or -2.7 - 72i.

Admittedly, our beautiful complex numbers were born from a long line of bad blood between 16th century mathematicians in Italy. But, honestly, that's hardly their fault; families are families. And what were they arguing about every holiday? Cubic equations. They could all solve quadratics ($ax^2 + bx + c = 0$; you know the song, I *know* you do), yet the cubic equation – the elusive $ax^3 + bx^2 + cx + d = 0$ – puzzled generations of scholars around the world for thousands of years.

Scipione del Ferro, a professor at the University of Bologna, is usually credited with solving cubics - or at least cubics without the x^2 term, called depressed cubics. Despite this stumping entire civilisations for thousands of years, despite all the glory he'd be showered in, he told nobody! Was he silly? Well, I don't think so - he solved the depressed cubic, after all. He kept his discovery a secret to keep his job; otherwise, some other mathematician could come along, challenge him to a maths duel (a really tricky maths test) and move into his office at the university! Del Ferro really liked Bologna! And if nobody else knew how to solve cubics, he could maintain his advantage! He kept this secret until he was on his literal deathbed in 1526, when he told his student Antonio Fior.

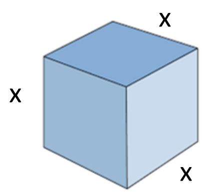
Fior was... less discreet. Though he never shared the actual solution, he wanted everybody (and their dog) to be aware he could solve depressed cubics. He went on to challenge the mathematician Niccolo Fontana (better known as Tartaglia, "the stammerer", due to an unfortunate encounter with a French soldier's sword) to ANOTHER maths duel. And of course, every single one of the questions Fior gave him was on depressed cubics.

So what did Tartaglia do? Did he cry? Maybe. Did he scream? It's possible. Did he surrender? No! He sat down and decided that if there really was a general solution, he could find it. And he did. Smart guy.

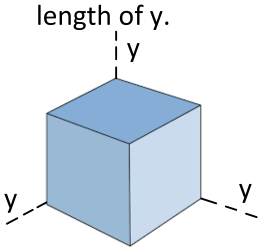
Nowadays, we'd solve cubic equations algebraically, but back in the day they didn't have the notation for it. Maths was used as a way to define and measure the world people could see around them, so they used a more tangible method – it's time for some geometry.

Take the depressed cubic equation $x^3 + x - 6 = 0$, which we'll rearrange to $x^3 + x = 6$ because honestly I don't know what a negative shape would look like.

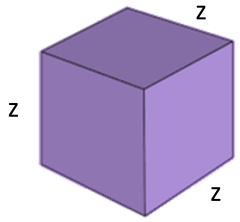
We can think of x^3 as being a cube with sides of length x. It has a volume x^3.



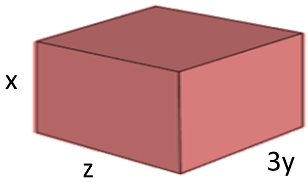
So to make this equal 6, we have to add a volume of x to the x^3. Let's imagine extending the sides of length x by a length of y.



Now, we get a larger cube. Let's say x + y = z, so this cube has sides of length z and a volume z^3.



Here's where we need to concentrate. If you picture that original x^3 cube placed inside the z^3 cube, there's some empty space as well, right? We can divide that empty space into several shapes. More specifically, there would be one cube with sides of length y, three cuboids with dimensions x by x by y, and three cuboids with dimensions x by y by y. Ignoring the little cube for a second (sorry little cube!), we can combine the cuboids to make – guess what – a bigger cuboid! This one has dimensions x by 3y by x+y (which is equal to z).



The volume of this cuboid is 3xyz = x (remember x from the original equation?), so now we get 3yz = 1 to make things simpler. Then, to go back to the original equation (adding in the little cube now!), we have $x^3 + x + y^3 = 6 + y^3$, which simplifies to $z^3 = 6 + y^3$.

I know I said no algebra, but let's do some sneaky simultaneous equations to make life easier.

$3yz = 1$

$z = \frac{1}{3y}$

$(\frac{1}{3y})^3 = 6 + y^3$

$\frac{1}{27y^3} = 6 + y^3$

$1 = 162y^3 + 27y^6$

Now, we turn this into a quadratic, which they already knew how to solve by completing the square.

$27(y^3)^2 + 162y^3 - 1 = 0$

$27[(y^3)^2 + 6y^3 - \frac{1}{27}] = 0$

$27[(y^3 + 3)^2 - 9 - \frac{1}{27}] = 0$

$27(y^3 + 3)^2 = 244$

$(y^3 + 3)^2 = \frac{244}{27}$

$y^3 = \sqrt{\frac{244}{27}} - 3$

$y^3 = 0.006166501882$

$y = 0.1833776021$

$z = \frac{1}{3y}$

$z = 1.817742895$

$x = z - y$

$\underline{x = 1.634365293}$

Whew! That was a lot of work. That really made me appreciate how difficult it was to be a 16th century mathematician, and I even cheated at the end by algebraically completing the square instead of doing it geometrically!

But there's something very interesting about this method: by solving the equation with shapes, you're only ever going to find positive real solutions. However, equations can have negative solutions and (wait for it…) complex solutions too; this one I've just done actually does have a complex pair of solutions, as well as the real one I found. Tartaglia's method does not allow for this as a possibility.

Still, it was a huge step in the right direction. Word quickly spread of Tartaglia's achievement, drawing attention from some of the greatest minds of the time. Another mathematician (just keep them coming) named Gerolamo Cardano used a variety of letters filled with flattery and/or insult to convince Tartaglia to visit him in Milan. Cardano sort of begged Tartaglia to show him the method to solve depressed cubics, and Tartaglia gave in, as long as he promised to keep it hush-hush. They made a secret oath and everything.

You can guess what comes next. As opposed to keeping it hush-hush, Cardano published the proof for the whole world to read in his book *Ars Magna*. However, to be fair, Cardano deserved some kudos; with his assistant Ludovico Ferrari, he'd managed to solve cubics generally! Not just the depressed ones! He found that, if you take an equation of the form $ax^3 + bx^2 + cx + d = 0$, and then substitute $(x - \frac{b}{3a})$ instead of x, then all the x^2 terms cancels out. Then, you can solve the equation using Tartaglia's method.

They'd done it!

So you can understand why Cardano was so excited to get this published – after all, he was primarily a physicist, meaning a failed maths duel couldn't ruin his life, which made the potential glory highly appealing. Besides, he later discovered that del Ferro (remember him?) had come up with Tartaglia's proof first, so he wasn't really stealing from *Tartaglia*... right?

Tartaglia was, very understandably, not happy. This all culminated in the ultimate maths duel smackdown / debate in Milan, between him and Ferrari. Tartaglia lost. I feel bad for the guy.

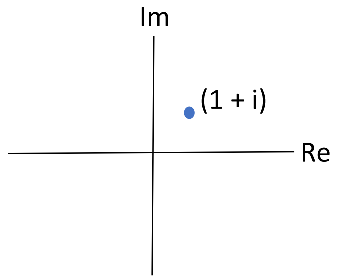
Sorry, I realise I've gone on about Renaissance mathematicians and their cubic-induced dramas for quite a while now. Complex numbers on the way!

In his book, Cardano very conveniently glossed over the fact that there *were* some cubic equations he couldn't solve because they gave him answers with the square roots of negative numbers. Sound familiar? He asked Tartaglia about this, but he was befuddled too. But, fear not, for yet another 16th century Italian mathematician was on the way! Rafael Bombelli was an engineer, and in his practical calculations of cubic equations he kept encountering the square roots of negative numbers. While generations of previous intellectuals had disregarded these entirely, Bombelli saw that they must have some real-world relevance, and he established some general rules for them in his 1572 book *L'Algebra*.

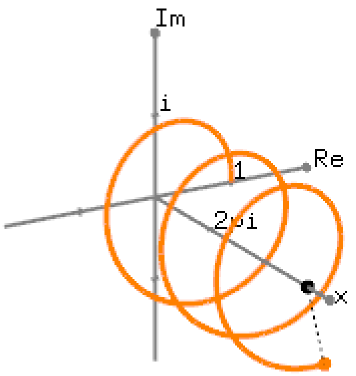
In the following centuries, more great mathematicians became interested in complex numbers. René Descartes actually coined the term 'imaginary'; Leonhard Euler was the first to use i to represent $\sqrt{-1}$. And speaking of Euler, let's talk about some real-life applications of complex numbers (if he were here right now, I'm sure Bombelli would be screaming, "Haha! I knew they'd be useful!").

Take the graph y = e^{ix}, very kindly brought to us by Euler. If you haven't met e before, don't panic, it's just a number: it's irrational, approximately 2.71828, and very handy when dealing with growth.

Now, you might be wondering: how on earth can we plot imaginary numbers on a graph? It might sound as far-fetched as trying to figure out what a negative shape looks like (that one hurts my brain). Lucky for us, there are special graphs called Argand diagrams to come to the rescue; where the x-axis would usually be is the real axis (basically the real number line), and perpendicular to it is the imaginary axis (where the y-axis would usually be).



I found a great diagram to show us what y = e^{ix} looks like in 3D, using an Argand diagram with an x-axis in there for the fun.



It's a spiral shape! e is the natural growth number, which gives the graph this continuous cycle quality… sort of like an alternating current! (I know it was on the tip of your tongue.) Compared to Cardano, the great mind who defeated the cubic equation but saw $\sqrt{-1}$ and ran for the hills, modern-day physicists regularly use Euler's work to analyse the behaviour of voltages and currents. This is essential to the field of electrical engineering.

Quick side note: $e^{ix} = \cos(x) + i\sin(x)$. The real part's the cosine curve and the imaginary part's the sine curve! People believed complex numbers were unnatural, but look! Trigonometry link! I love it.

So to conclude this long and rambling discussion, complex numbers appeared where they were least expected. It hasn't even been 500 years since mathematicians were using shapes to solve equations (can verify this is tricky), using methods which didn't always give every solution. And there complex numbers were, ignored and rejected, until people built on the work of those before them, thought outside the box, and innovated new ideas. Who knows? Without Tartaglia's solution to depressed cubics, maybe Bombelli could never have published *L'Algebra*, and we'd never have met y = e^{ix}, and then we couldn't charge our mobile phones! The horror!

But seriously, it's nice how this stuff turns out, and it shows how important it is to be creative. So while complex numbers might not be as simple as I promised, hopefully they're not quite as scary and mysterious as you thought. And hey, at least you can impress your friends with facts about Renaissance mathematicians! Cool, right? (Maybe not.)

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