Bivectors: Better imaginary numbers, with a twist.

The title of this poem is quite a bold claim.

Some may view it as insane.

Others ask why should complex numbers be defamed.

Simply read this,

and allow me to attempt to convince,

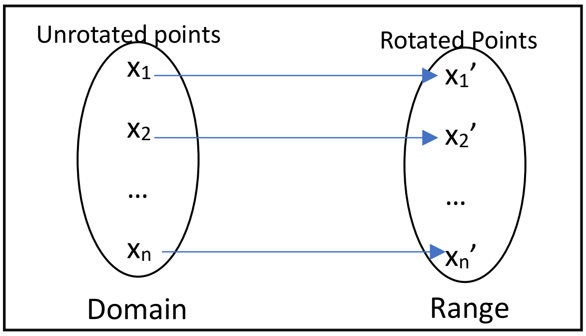
why it could be invaluable,

for a mathematician's genesis.

The above introductory poem
Pertains to a particular pedagogic importance,
That something called 'bivectors' could possess.
And it's this I'd like to stress.
If they were given a chance,
They could serve as aids for introducing complex numbers,
Through a less 'hand-wavey' stance.

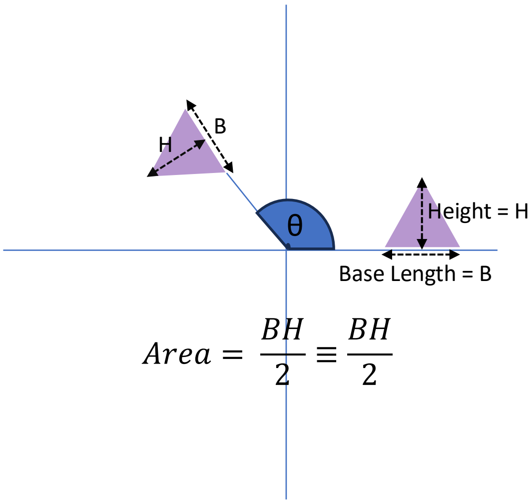
1. Winning with spinning:

Rotations occur in flat planes.
They are functions
As the initial points,
Are the domain.



With the co-ordinates after the transformation being a range.
They are important for many reasons,
Especially for their lack of change.
Where the area of a given shape,
Or length of a vector,
Is left un-mangled.
Despite this, the output can be different due to the angle.
An angle which, around a given axis, it rotates.
Now I'll tell you why spinning is great.

Spinning can demonstrate a very powerful symmetry,
What I mean by this is that the properties,
Of the given 'rotatee',
Are preserved,
Mathematicians call them 'isometries'.
Meaning that it's their mission,
To have lengths that are left in mint condition,
Despite having changed position.



$$Area = \frac{BH}{2} \equiv \frac{BH}{2}$$

A physicist for example, could find this quite useful,
Einstein stated,
In one of the theories for which he is famed,
'The laws of physics are constant,
In all reference frames'.
(One of the postulates for special relativity),
They are an invariant property,
Otherwise referred to as a symmetry.

2. Vectors and their methods of multiplication:

Vectors can be introduced to students in the realm of physics,

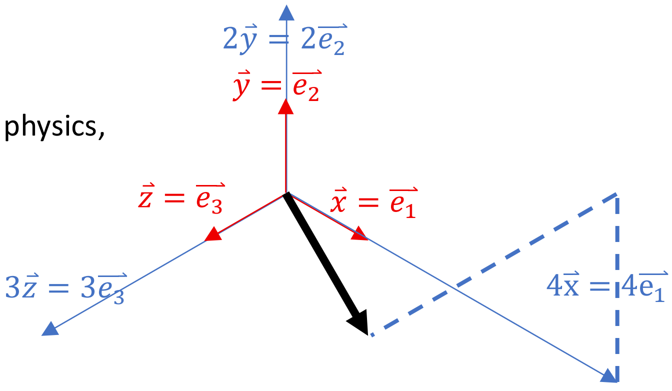
In terms of direction and magnitude.

Take F=ma,

Given the correct attitude

This linchpin of mechanics, (in the Newtonian sector),

Can be considered a connector between two vectors.



$$\vec{v} = 4\vec{x} + 2\vec{y} + 3\vec{z} = 4\overrightarrow{e_1} + 2\overrightarrow{e_2} + 3\overrightarrow{e_3}$$

This is how a vector is typically described, with each basis vector $(\vec{x}, \vec{y}, \vec{z})$ being a step in the respective directions. We will use the $(\overrightarrow{e_1}, \overrightarrow{e_2}, \overrightarrow{e_3})$ instead, for ease of notation.

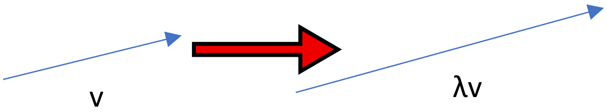
It is simple enough to understand,

From this example grand.

How scalars interact with vectors,

They, as the term suggests, scale the magnitude of said vector,

Making it bigger and better.



Despite this,

The product of two vectors is open to interpretation.

Note: Scalars are just numbers

From dot products,

To cross products, to matrix multiplication,[1]

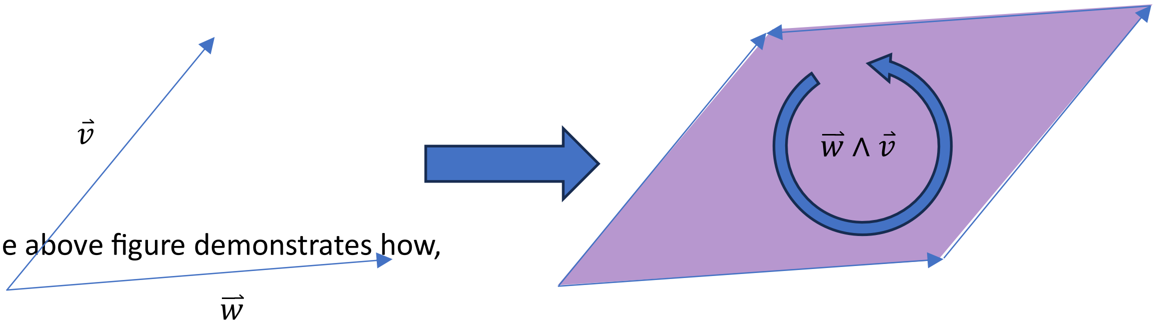
This is all dependent on the situation and one's education.

My way, is the exterior (or wedge) product. This operation takes two vectors, and outputs a mathematical object known as a bivector.

They are such wonders.

It can be represented as is shown under,

(This is useful to explain imaginary numbers.)



The above figure demonstrates how,

The concepts of vectors and bivectors share similar characteristics.

Although this connection isn't necessarily holistic.

We shall expand on this notion of parallelograms and twists,

And by doing so make the concept of bivectors less mystic.

(The direction is analogous to the clockwise or anti-clockwise loop, and magnitude is represented through the area)

$$v \times v' = (a\vec{e_1} + b\vec{e_2} + c\vec{e_3}) \times (d\vec{e_1} + e\vec{e_2} + f\vec{e_3}) = (bf - ce)\vec{e_1} - (af - cd)\vec{e_2} + (ae - bd)\vec{e_3}$$

3. The Wedge Product:

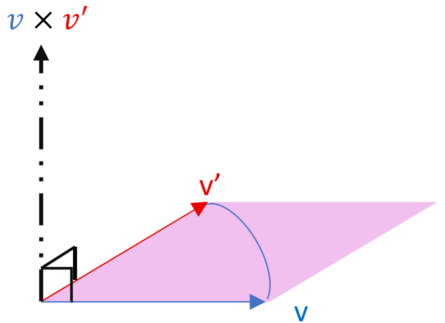
Traditionally when vectors are rotated,

One uses a cross-product,

Where in which a vector's future is stated.

With the initial vector being to the left of the cross,

And on its right is where its final position is fated.



Here, v is the original vector, and v' is the rotated vector, the product of this operation is the perpendicular vector in black.

It is unfortunate then,

For the following is true of this operation.

It is limited only to the third (and strangely the seventh) dimension.

One reason for this constraint,

Is due to the orthogonal pseudovector

About which our vector rotates.

Instead for us,

We shall replace this sideways plus (x),

With a symbol used earlier.

The wedge (∧), it is similar in outcome to a cross product,

Except it generalises to all dimensions, Which is how the former operation obstructs.

As already mentioned, the cross has a vector orthogonal to both inputs,

Being a byproduct.

The Wedges Key Properties:

- $\vec{w} \wedge \vec{v} = -(\vec{v} \wedge \vec{w})$ (Anti-commutativity)
- $\vec{w} \wedge \vec{w} = 0$ (Any vector wedged with itself is zero)

There are more properties and given identities,

But these are the two that shall be needed currently.

The reason for the bivector's anti-commutativity,

Is due to the earlier explanation,

That the bivectors direction,

Is in proportion to the shown chirality. (↺= -↻)

As for the self-wedging property?

The magnitude of a bivector is its area, in a sense,

Wedging itself yields no width, only length,

Thus, the area is zero.

4. Unit bivectors:

Like regular vectors,

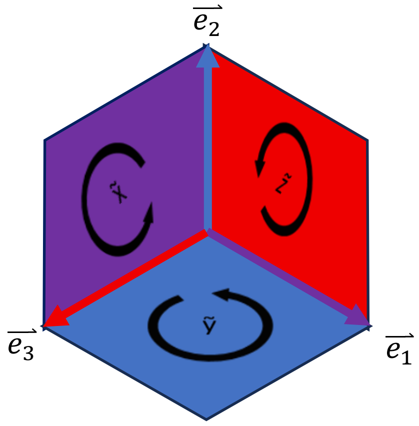
Bivectors also have a bases.

For three dimensions, they are $\overrightarrow{e_1} \wedge \overrightarrow{e_2}$=z̃, $\overrightarrow{e_2} \wedge \overrightarrow{e_3}$=x̃, and $\overrightarrow{e_3} \wedge \overrightarrow{e_1}$=ỹ.

They are similar to unit vectors of vector spaces,

Meaning that in a bivector's description,

You would find the bases traces.



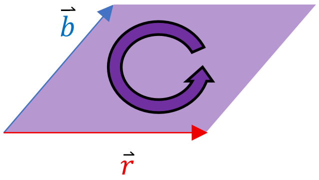
The Unit Bivectors in three dimensions

- Example in two dimensions:

$\vec{r} = (a\overrightarrow{\boldsymbol{e_1}} + b\overrightarrow{\boldsymbol{e_2}})$,

$\vec{b} = (c\overrightarrow{\boldsymbol{e_1}} + d\overrightarrow{\boldsymbol{e_2}})$

$\vec{r} \wedge \vec{b} = A$



$$A = \ (a\vec{\boldsymbol{e_1}} + b\vec{\boldsymbol{e_2}}) \wedge \ (c\vec{\boldsymbol{e_1}} + d\vec{\boldsymbol{e_2}})$$

$$A = ac(\vec{\boldsymbol{e_1}} \wedge \vec{\boldsymbol{e_1}}) + ad(\vec{\boldsymbol{e_1}} \wedge \ \vec{\boldsymbol{e_2}}) + bc(\vec{\boldsymbol{e_2}} \wedge \ \vec{\boldsymbol{e_1}}) + bd\ (\vec{\boldsymbol{e_2}} \wedge \ \vec{\boldsymbol{e_2}})$$

Which becomes:

$$A = ad - bc(\vec{\boldsymbol{e_1}} \wedge \ \vec{\boldsymbol{e_2}})$$

The determinant?! Isn't that cool.

We seem to have derived a new meaning for the determinant,

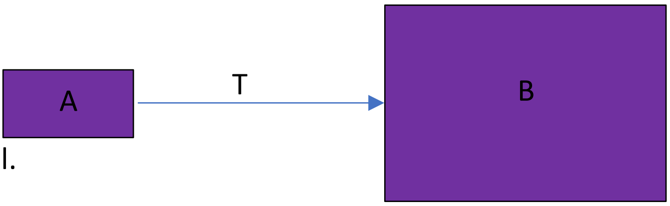
By accident!

How exciting!

What with it being made with this wedging tool.

Transformation: $T \times A = B$

$$Determinant\ of\ T = \frac{|B|}{|A|}$$



Note: In Linear Algebra, the determinant can be thought of as the amount by which a given area can change underneath a general linear mapping [meaning that it's a transformation that keeps straight lines straight.]

- <u>Example in three dimensions:</u>

But the fun doesn't end at two.

Look with glee,

At the general result for working this out,

With dimensions three.

$$\vec{r} \wedge \vec{b} = (a\vec{\boldsymbol{e_1}} + b\vec{\boldsymbol{e_2}} + c\vec{\boldsymbol{e_3}}) \wedge \ (d\vec{\boldsymbol{e_1}} + e\vec{\boldsymbol{e_2}} + f\vec{\boldsymbol{e_3}})$$

The equation beneath, can, from the above is changed.

And using the previous result from earlier, as an example,

Hopefully leads to this alteration seeming un-strange.

$$\begin{aligned}\vec{r} \wedge \vec{b} = {} & (ad)(\vec{\boldsymbol{e_1}} \wedge \ \vec{\boldsymbol{e_1}}) + (ae)(\vec{\boldsymbol{e_1}} \wedge \ \vec{\boldsymbol{e_2}}) + (af)(\vec{\boldsymbol{e_1}} \wedge \vec{\boldsymbol{e_3}}) + (bd)(\vec{\boldsymbol{e_2}} \wedge \ \vec{\boldsymbol{e_1}}) \\ & + (be)(\vec{\boldsymbol{e_2}} \wedge \vec{\boldsymbol{e_2}}) + (bf)(\vec{\boldsymbol{e_2}} \wedge \vec{\boldsymbol{e_3}}) + (cd)(\vec{\boldsymbol{e_3}} \wedge \vec{\boldsymbol{e_1}}) + (ce)(\vec{\boldsymbol{e_3}} \wedge \vec{\boldsymbol{e_2}}) \\ & + (cf)(\vec{\boldsymbol{e_3}} \wedge \vec{\boldsymbol{e_3}}) =\end{aligned}$$

$$\vec{r} \wedge \vec{b} = (bf - ce)(\vec{\boldsymbol{e_2}} \wedge \ \vec{\boldsymbol{e_3}}) - (af - cd)(\vec{\boldsymbol{e_3}} \wedge \ \vec{\boldsymbol{e_1}}) + (ae - bd)(\vec{\boldsymbol{e_1}} \wedge \ \vec{\boldsymbol{e_2}})$$

This equation should ring a bell.

It is the cross-product formula,

From earlier!

And now, instead of being a recipe delivered from up high,

Which causes students to be insanely driven.

It's instead logically derived.

But what's more,

Reason is given.

Which helps those who ask 'why?'.

(Note: I cunningly flipped the middle bivector component in order to introduce a negative so the connection pops out more.)

5. The Geometric Product:

There is also a geometric product,

But we shan't linger long on it,

(Despite how it's an amazingly cool a piece of kit).

It allows for many things,

Least of all convenience of notation,

When operating with unit bivectors.

Note: The dot product is used here, a formula for which is:

$$\vec{r}\cdot\vec{b} = |\vec{r}||\vec{b}|\cos(\theta) = \textit{The length of one vector} \times \textit{The length of another} \times \textit{Distance between tip} = a\ (\textit{Some number})$$

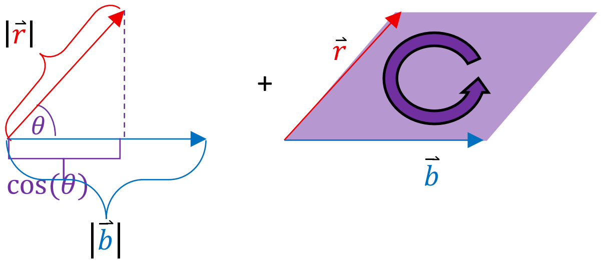
Here is this beautiful method of multiplication:

$$\vec{r}\vec{b} = \vec{r}\cdot\vec{b} + \vec{r}\wedge\vec{b} = a + c\circlearrowleft$$

A scalar number,

Summed with something to do with rotation.

(z=a+bi, hint hint.)



6. A daring squaring:

Let us now, very teasingly, define the square of a unit bivector as such:

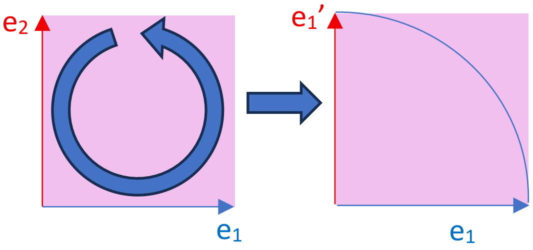
$$(e_1e_2)^2 = i^2$$

From this we can derive:

$$(e_1e_2e_1e_2) = i^2$$

Which then becomes, using anti-commutativity:

$$-(e_2e_1e_1e_2) = i^2$$



And then it becomes, using the self-wedging property:

$$-(e_2(e_1 \cdot e_1 + e_1 \wedge e_1)e_2) = -(e_2(1+0)e_2) = i^2$$

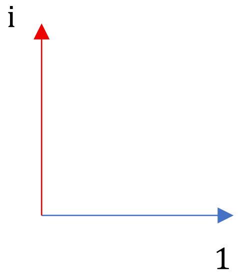
Voila!

$$-(e_2(1)e_2) = -(e_2e_2)(1) = -(e_2 \cdot e_2 + e_2 \wedge e_2)(1) = -(1+0)(1) = -1 = i^2$$

This demonstrates that rotating something by 90 degrees,

Should be the same as the product of a number and something,

Quite ‘imaginary’ you see.



How cool is that!

There is now a more concrete intuition,

That is furthermore,

Logically driven!

And for the sceptical student.

An explanation excellent,

Has been given,

With valid reason,

Without hand-wavey claims,

Which are rather lame.

7. Discussion, Conclusion and Connections:

If you accept the above as logically true,

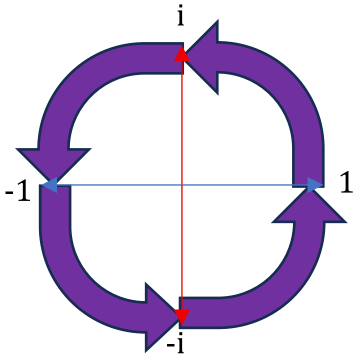
Then there are other things that can be shown too,

For example, it follows obviously that.

$$-1 \times 1 = -1$$

However, by using the rules of negative multiplication,

$$i^3 = i^2 \times i = -1 \times i = -i$$

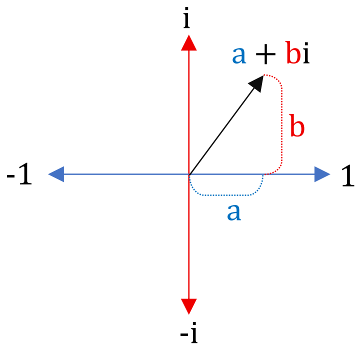


Gadzooks! It loops!

If negative **i** is likewise treated as negative one,

(Meaning that it faces the other way from its positive counterpart),

It seems as though a basis of vectors can be summed.
With an ‘imaginary’ ending (b) and real start (a).
These are the rotations as from before,
They have the same general structure,
Of which I absolutely adore.
And what’s more,
They can be extended onwards,
To higher dimensions galore.

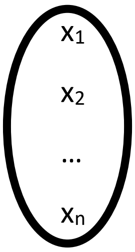


This way i is a 90° turn. ($a + c$ ↺).
This set of numbers represented as
two axes called ‘The Complex Plane’

These geometric products, bivectors, and loops,
Can be a stepping stone to learning,
What mathematicians refer to as groups.

Roughly, a group is a set,
Equipped with a ‘binary operation’,
This is jargon,
From a mathematician.

The humble set, AKA a bunch of stuff



More simply,
A set is a metaphorical bag of many a mathematical object,
Numbers, letters, bivectors,
This is what a mathematician would call a ‘set’.

Additionally, a binary operation is an action,
With two elements of this mathematical purse.
Where the outcome of the operation,
Has the added curse.
That the result isn’t an element out of the pouch.

It turns out that the structure that's been described so far,

Can be extended to form a group,

Called Hamiltonians,

(AKA quaternions).

Hamiltonians are my favourite group,

Lewis Caroll[2] wouldn't agree,

He thought these notions of i, j, k,

All squaring to -1,

Rather silly.

But I refute this claim,

I think quaternions,

Are to a mathematician's gain.

This symbol means "has the same pattern as"

→∧↓	1	e_1	e_2	e_3
1	1	e_1	e_2	e_3
e_1	e_1	-1	$-e_3$	e_2
e_2	e_2	e_3	-1	$-e_1$
e_3	e_3	$-e_2$	e_1	-1

$\cong$

→×↓	1	i	j	k
1	1	i	j	k
i	i	-1	-k	j
j	j	k	-1	-i
k	k	-j	i	-1

8. An alternative path for the alternative path to complex numbers:

There is another formula,

For the wedge.

That is similar to the dot. ($|\vec{a}||\vec{b}|\cos\theta$)

It isn't a lot:

This is related to the area of a parallelogram:

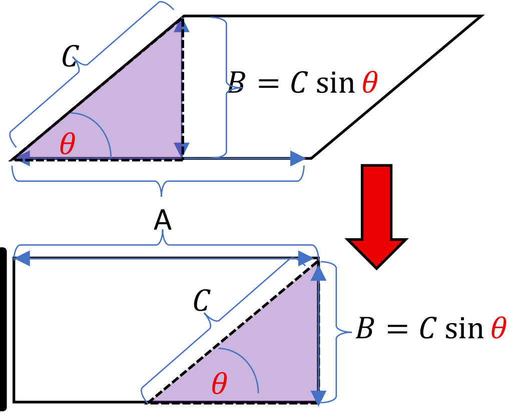
$$Area = A \times B = A \times C \times \sin\theta$$

Which is proven geometrically:

$$(\vec{a} \wedge \vec{b})^2 = -|\vec{a}|^2|\vec{b}|^2 \sin^2(\theta)$$

$$\vec{a} \wedge \vec{b} = \sqrt{-|\vec{a}|^2|\vec{b}|^2 \sin^2(\theta)} = |\vec{a}||\vec{b}| i \sin\theta$$

If we use the geometric product from before,



This works since the area in purple can be moved. (Becoming a rectangle which is easier to calculate)

One could thus perform the following chore:

$$\vec{a}\vec{b} = \vec{a} \cdot \vec{b} + \vec{a} \wedge \vec{b}$$

$$\vec{a}\vec{b} = |\vec{a}||\vec{b}| \cos\theta + |\vec{a}||\vec{b}| i \sin\theta$$

If we represent these scores,

On the complex plane from before

We arrive at:

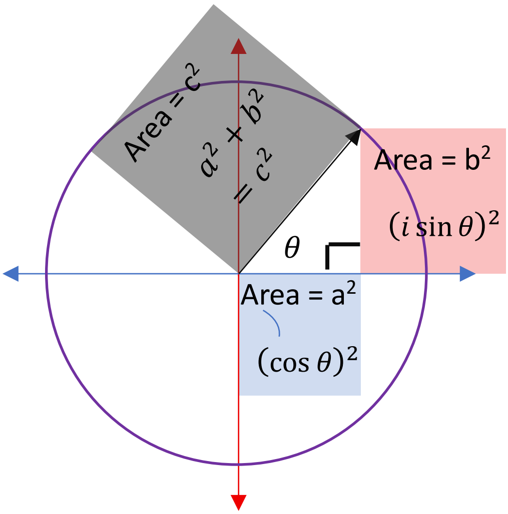
$$\vec{a}\vec{b} = |\vec{a}||\vec{b}|(\cos\theta + i\sin\theta)$$

And then by using a theorem ancient,

$$\vec{a}\vec{b}^2 = |\vec{a}|^2|\vec{b}|^2(\cos^2\theta + i\sin^2\theta)$$

On can derive a visualisation,

For these abstract machinations.



The reason a circle is also present is to demonstrate how as θ increases, the tip of the arrow remains a fixed distance as the tilt changes. The same way compasses draw circles.

And furthermore

The equation from before,

$$|\vec{a}||\vec{b}|(\cos\theta + i\sin\theta) = \vec{a}\vec{b}$$

Is ANOTHER way to write complex numbers,

Giving further implication

Of their very close relation.

Thank you for reading this poem upon one way of reading the Nature's poetry.

(What most people call mathematics)

Footnotes:

[1] (This counts as a form of vector multiplication as the list of linear transformations from one vector space two another itself satisfies the properties of a vector space. Consider it as increasingly abstract ways to look at a rotation. One property is associativity, as an example, if you rotate yourself ϕ degrees in an anti-clockwise fashion, and ζ degrees in a clockwise fashion, it's the same as rotating ζ degrees clockwise and then ϕ degrees anti-

clockwise. Sort of like adding one-dimensional vectors to one another. [This example also demonstrates the idea of an inverse transformation.])

[2] (Yes, Lewis Caroll of ‘Alice and Wonderland’. He was also an Oxford educated mathematician he didn’t think spinning was great, so he cruelly mocked it in the scene where the Mad Hatter has a tea party with Alice.)