Complex Quadratics, Rooting The Roots

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April 2026

1 Introduction

Almost all people are familiar with the natural numbers, most are too with the rationals, and some even the irrationals (collectively the reals). Fewer yet are yet aware of complex numbers a often hard to accept but brilliantly important part of numbers.

Most people who know much algebra are familiar with quadratics, and how to solve them. These are usually written in the form $ax^2 + bx + c$ with a, b, c being real numbers, and so from these two concepts, which are already very interlinked as we will see, seems only natural and so we shall.

2 Complex Numbers and Their Justification

Their are several different reasons for accepting the complex numbers however, the best argument may well be their usefulness. The complex numbers are in the form $a + bi$ where $a, b \in \mathbb{R}$ (a and b are real numbers) and $i = \sqrt{-1}$ (the imaginary unit) what this allows us to do is take square roots of negative number, for example:

$$\sqrt{-9} = \sqrt{9} \times \sqrt{-1} = 3i$$
$$\sqrt{-16} = \sqrt{16} \times \sqrt{-1} = 4i$$

Beyond this, complex numbers have great many uses in the sciences, due to their nature of encoding oscillation. In the purely mathematical sense this can be easily seen in things like Euler's formula and the complex forms of trigonometric functions:

$$a + bi = re^{i\theta}$$
$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$
$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

These will not formally proved here (due to extensive resources pertaining to it). Returning to the scientific side of things, the Laplace transform, used to make working with differential equations much easier fundamentally operates with complex numbers.

Beyond this, the complex numbers are algebraically closed, meaning using these operations: $\sqrt[n]{}, \times, \div, +, -$. The reals do not have this property under $\sqrt[2n-1]{}$. This has the effect of making every polynomial ($a_1x^n + a_2x^{n-2} + a_3x^{n-2} + ...a_nx + a_{n+1}$) have n solutions, a result required by the fundamental law of algebra.

3 Quadratics

For quadratics with real coefficients (ax^2+bx+c) we can always find both roots using the quadratic formula $x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$, sometimes (when $b^2-4ac<0$) we will get two complex roots. The quadratic formula is derived used a process known as completing the square, the derivation goes as follows:

$$ax^2+bx+c=0$$
$$x^2+\frac{b}{a}x=\frac{-c}{a}$$
$$x^2+\frac{b}{a}x-(\frac{b}{2a})^2=\frac{-c}{a}-(\frac{b}{2a})^2$$
$$(x+\frac{b}{2a})^2=\frac{-c}{a}-(\frac{b}{2a})^2$$
$$(x+\frac{b}{2a})^2=\frac{-4ac}{4a^2}-(\frac{b^2}{4a^2})$$
$$(x+\frac{b}{2a})^2=\frac{b^2-4ac}{4a^2}$$
$$x+\frac{b}{2a}=\pm\sqrt{\frac{b^2-4ac}{4a^2}}$$
$$x=\pm\frac{\sqrt{b^2-4ac}}{2a}-\frac{b}{2a}$$
$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

Ultimately all of these steps do work for complex numbers for this reason we can rewrite our formula to represent this: $x = \frac{-z_2\pm\sqrt{z_2^2-4z_1z_3}}{2z_1}$ where w, v, z are complex numbers. Using this we can now study the movement of these roots on the complex plane compared to those with real coefficients. Setting $z_1 = 3+2i, z_2 = -1-i, z_3 = 2-i$ we can, using the quadratic formula and some calculations find our roots to be $0.60727517...+0.70508539...i$ and $-0.22265979...-0.62816231...i$ which, surprisingly can be acquired to exact values (found thanks to WolframAlpha):

$$x_1 = -\frac{1}{26}-\frac{1}{13}\sqrt{2}\sqrt[4]{265}\sin\left(\frac{1}{2}(\tan^{-1}\left(\frac{3}{16}\right)-\pi)\right)-\frac{3\sqrt[4]{265}\cos\left(\frac{1}{2}(\tan^{-1}\left(\frac{3}{16}\right)-\pi)\right)}{13\sqrt{2}}+$$
$$i\left(\frac{5}{26}-\frac{3\sqrt[4]{265}\sin\left(\frac{1}{2}(\tan^{-1}\left(\frac{3}{16}\right)-\pi)\right)}{13\sqrt{2}}+\frac{1}{13}\sqrt{2}\sqrt[4]{265}\cos\left(\frac{1}{2}(\tan^{-1}\left(\frac{3}{16}\right)-\pi)\right)\right)$$
$$x_2 = -\frac{1}{26}+\frac{1}{13}\sqrt{2}\sqrt[4]{265}\sin\left(\frac{1}{2}(\tan^{-1}\left(\frac{3}{16}\right)-\pi)\right)+\frac{3\sqrt[4]{265}\cos\left(\frac{1}{2}(\tan^{-1}\left(\frac{3}{16}\right)-\pi)\right)}{13\sqrt{2}}+$$
$$i\left(\frac{5}{26}+\frac{3\sqrt[4]{265}\sin\left(\frac{1}{2}(\tan^{-1}\left(\frac{3}{16}\right)-\pi)\right)}{13\sqrt{2}}-\frac{1}{13}\sqrt{2}\sqrt[4]{265}\cos\left(\frac{1}{2}(\tan^{-1}\left(\frac{3}{16}\right)-\pi)\right)\right)$$

The reason for the intense complexity, and trigonometric functions, compared to complex solutions to regular quadratics, is where the complex parts arise from. In a regular quadratic only the square root can result in imaginary parts, whereas in a complex quadratic all parts can and will result in imaginary parts. The trigonometric functions arise from the square root, which to complex numbers is equivalent to square-rooting the magnitude of the number, and halving the angle.

3.1 Defining The Complex Square Root

While the prior given explanation for the the square root may make sense given its alignment, it is best to more rigorous prove it. As earlier stated a complex number $a + bi$ can be rewritten in the form $re^{i\theta}$ where $r = \sqrt{a^2 + b^2}$ and $\theta = \tan^{-1}\left(\frac{b}{a}\right)$. from here we can rewrite $\sqrt{a + bi}$ as $r^{\frac{1}{2}} e^{i\theta^{\frac{1}{2}}}$. Thus proving the earlier stated way of square rooting a complex number. When returning this to the a+bi form of complex numbers we can understand why in the earlier example we see $\cos(\tan^{-1}(\frac{b}{a}))$ and $\sin(\tan^{-1}(\frac{b}{a}))$.

3.2 The True Complex Quadratic Formula

Looking at a few more, complex quadratics we can begin to see a pattern in their structure (which will not be written out due to their size), suggesting the existence of a more processed complex quadratic formula than the aforementioned one.

Our formula should be able to isolate the real and imaginary parts of the roots of our quadratic this is so that we can easily isolate them simply by placing. After some algebra (involving some extra tricks to perform division) we can derive the formula (this will not be done here for brevity as the full derivation is very lengthy):

$$x = \frac{-2a_1b_2 + 2a_2b_1}{4a_1^2 + 4b_1^2} \pm (\frac{4a_1\sqrt[4]{a_1^2a_3^2 + a_1a_3b_1b_3 + b_1^2b_3^2 + a_1^2b_3^2 + a_3^2b_1^2}}{4a_1^2 + 4b_1^2}\cos(\tan^{-1}\frac{a_1b_3 + a_3b_1}{b_1b_2 - a_1a_3}) +$$

$$\frac{-2b_1 4a_1\sqrt[4]{a_1^2a_3^2 + a_1a_3b_1b_3 + b_1^2b_3^2 + a_1^2b_3^2 + a_3^2b_1^2}}{4a_1^2 + 4b_1^2}\sin(\tan^{-1}\frac{a_1b_3 + a_3b_1}{b_1b_2 - a_1a_3})) + \frac{-2a_1b_2 + 2a_2b_1}{4a_1^2 + 4b_1^2} \pm$$

$$(\frac{4a_1\sqrt[4]{a_1^2a_3^2 + a_1a_3b_1b_3 + b_1^2b_3^2 + a_1^2b_3^2 + a_3^2b_1^2}}{4a_1^2 + 4b_1^2}\sin(\tan^{-1}\frac{a_1b_3 + a_3b_1}{b_1b_2 - a_1a_3}) +$$

$$\frac{-2b_1 4a_1\sqrt[4]{a_1^2a_3^2 + a_1a_3b_1b_3 + b_1^2b_3^2 + a_1^2b_3^2 + a_3^2b_1^2}}{4a_1^2 + 4b_1^2}\cos(\tan^{-1}\frac{a_1b_3 + a_3b_1}{b_1b_2 - a_1a_3}))i$$

apologies for any errors in this formula it took forever to derive and I don't have time to check it

Interestingly this formula does not imply the connection to any form of symmetry in the roots like in the real quadratic formula.

4 Conclusion

While the usefulness of complex quadratics remains to be seen, undoubtedly they pose as interesting mathematical objects, that can help to demonstrate how to work with complex numbers. I appreciate you reading this paper, and hope you learnt something, found it interesting, or it made you think about something new.