# Complexity from simplicity

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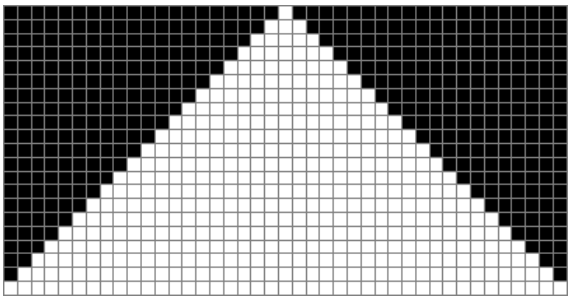
## Introduction: The flickering lights of Elemento

In the sprawling kingdom of Elemento, nobody ruled. Neither a council, nor a king, but instead, rules that its citizens must follow. Each terrace of infinite houses had their own single rule, established on the day the kingdom was founded, which governed the lantern of a resident in accordance to the lanterns of their neighbors. The rule[1] of a certain terrace decrees:

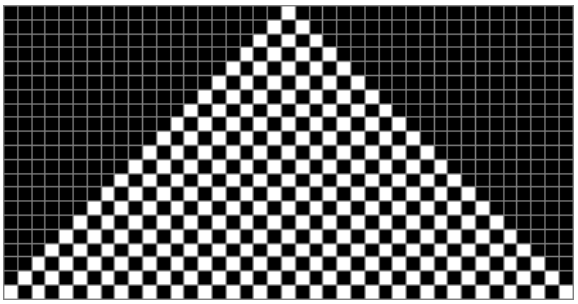
*"If the lantern of you or one of your neighbors is lit, light your lantern."*



At first, when these laws were enacted, the house in the middle of each terrace lit their lantern, and for every dawn, a new pattern of lit and unlit lanterns would emerge. For this specific rule, it would eventually light the lanterns of every resident in its corresponding terrace, creating a uniform structure that is predictable when viewed across time.

Another law[2] of a nearby terrace, four blocks down the previous, states:

*"If the lantern of one of your neighbors is lit, light your lantern; otherwise, extinguish it."*



Unlike the previous rule, the rule ignores the state of your lantern and tells you when to extinguish it. Instead of a uniformly growing expanse of lit lanterns, we will see one with lanterns alternating between unlit and lit states. When viewed across time, we will see a repetitive checkerboard pattern.

From the two rules we viewed, we can assume that the light of lanterns that occur in each terrace when viewed through time is simple and easy to predict. This is probably because of the simplicity of the rules that only say when to light or extinguish a lantern. However, consider the rule[3] of the terrace about two-thirds of the way down from the previous to the first terrace:

*"If the lantern of a neighbor is lit, and the lantern of the other neighbor is unlit, light your lantern; otherwise, extinguish it."*

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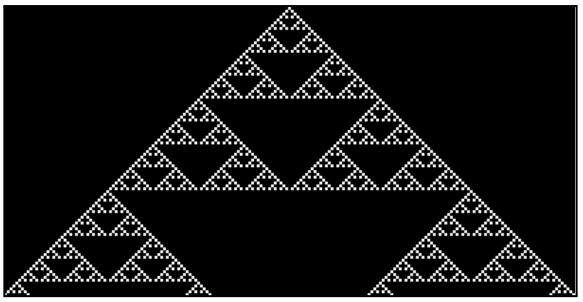
[1] This will be rule #254 in the rule numbering system mentioned later in this essay.
[2] This will be rule #250 in the rule numbering system mentioned later in this essay.
[3] This will be rule #90 in the rule numbering system mentioned later in this essay.

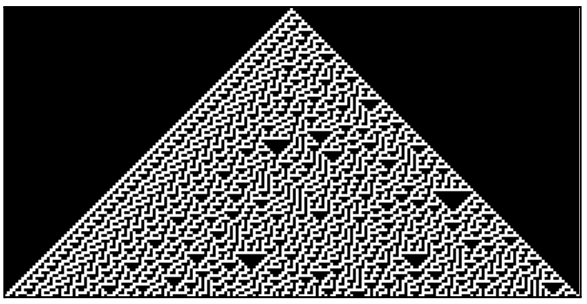
This time, the states of each lantern matter with each other. Viewing the terrace across a long amount of time, a more complex and intricate fractal structure emerges, composed of nested triangles.



Now, instead, we can assume the rules have patterns that form regular structures, either having uniform, repetitive, or nested structures; though, select rules can have more complicated and irregular structures, despite the simple nature and starting condition. Consider the following rule[4] and pattern of the terrace, again two-thirds of the way down:

*"If the lanterns of you and your right-hand neighbor are lit, light or extinguish the lantern to the state of the lantern of your left-hand neighbor; otherwise, do the opposite."*

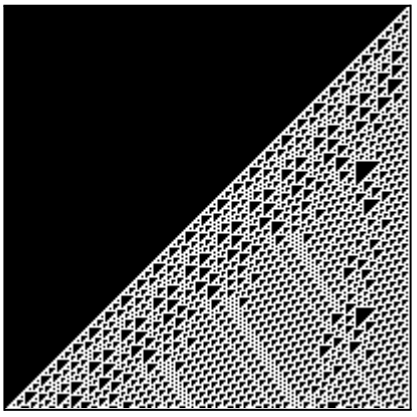
What emerges is greatly complicated with little regularity and is an example of a fundamental moral instilled in the kingdom, that even if one's rule has simple and elementary underlying conditions, the pattern that emerges from it can be highly sophisticated and erratic in nature. No such order may be derived from it, even with the most advanced methods of analysis. However, small structures such as the ever expanding triangular bands on the left side of the image can be pointed out.



So, one thing is for sure, the residents of the kingdom believe that complexity can be derived from simplicity, but what about both? Consider the rule[5] below:

*"If both your neighbor's lanterns are either lit or unlit, unless your lantern is unlit, and the lantern of your left-hand neighbor is lit, light your lantern; otherwise, extinguish it."*

What emerges is an unexpected mixture of localized structures, those on the left hand side being repetitive and those on the right hand side a less regular and essentially random pattern showing regularity and irregularity throughout. These structures, by themselves, can be considered simple, but the way how they interact with each other is unnecessarily convoluted, making it impossible to predict. For example, after much time, the beams that form constantly hit a group of triangle structures near the left hand side, setting off more complex behavior.

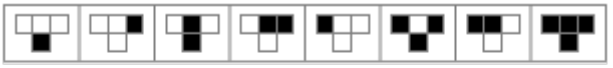


[4] This will be rule #30 in the rule numbering system mentioned later in this essay.
[5] This will be rule #110 in the rule numbering system mentioned later in this essay.

Based on these rules, there are four types of emergent behavior that originate from the rules of the terraces with a single lantern lit. The first are uniform, repetitive, and predictable patterns, that being the first two rules mentioned here. The second is a nested and fractal structure that is still regular in terms of the other rules. No regularity is now seen in the third type, having a pattern that is erratic and essentially random, and both regularity and irregularity is seen in the last type, having intricate structures that interact collectively in complicated ways. It is a bit surprising how complicated and diverse a simple tradition among the residents of Elemento can be, considering their simple and direct rule sets.

**Binary lighting patterns**

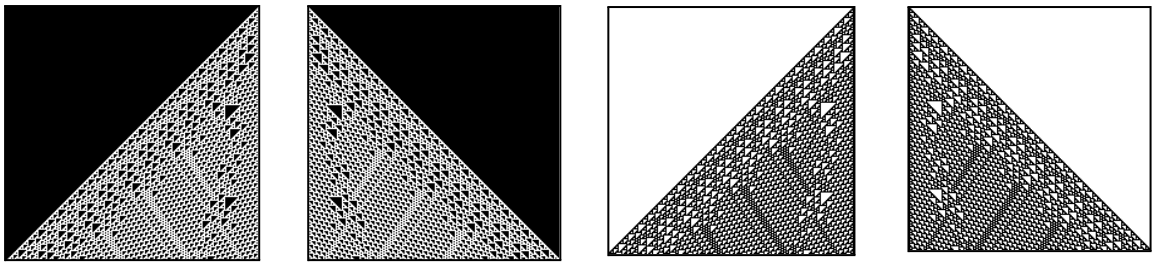
Since the structure of the rules of each terrace are generally the same, we can create a accessible visual representation of the rules that determines the state of a house's lantern according to the states of it and its neighbor lanterns the day before. The top row shows the 8 possible configurations of 3 lit or unlit lanterns, and the bottom row dictates the outcome of the middle lantern the next day.



Rule 90

Additionally, given 8 possible configurations and 2 resulting states, there will be $2^8$, or 256 possible rules for each terrace in the kingdom. We can number these rules by thinking of each resulting state as a binary representation of some number from 0 to 255. We can do this by ordering these from lit to unlit configurations and interpreting the resulting states as a binary number that names the rule and its corresponding terrace as well.

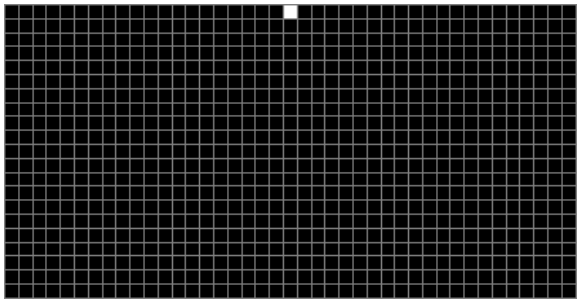
However, some of these rules, when represented as a bit map through time, seem identical to other rules at most, with the pattern either being mirrored, inverted, or both. For example, #110, the previous rule, has a mirrored rule of #124, an inverted rule of #137, and a mirrored and inverted rule of #193. Some rules, when altered through either transformation, are still the same. In total, there are 88 rules that are unique to themselves regardless of any transformation.



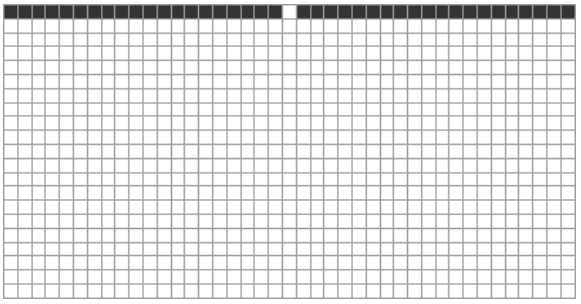
Rule #110 Rule #124 Rule #137 Rule #193

With this numbering scheme, let us now analyze how the lanterns flicker across specific terraces, and if applicable, the patterns of the numbers in their patterns' binary representations.

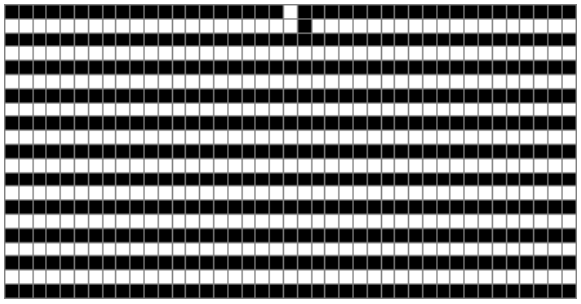
The simplest of the terraces are the ones that become forever dark or light after a single day, such as terrace #0 where every lantern dies or terrace #255 where every lantern stays on. Some of these predictable terraces also alternate day by day between dark and light, like terrace #7 and terrace #127. There are also terraces that leave a noticeable continuous trail of lanterns that stand out from the background lighting. Some trails remain in their place, like terrace #4, or can move left or right, like #103.



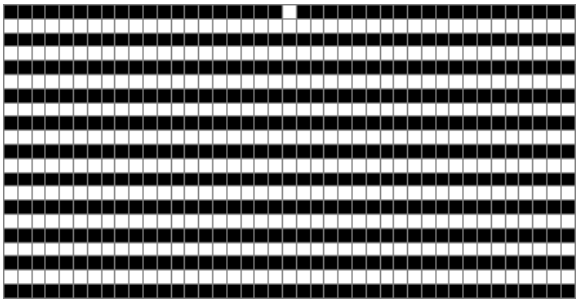
Rule #0



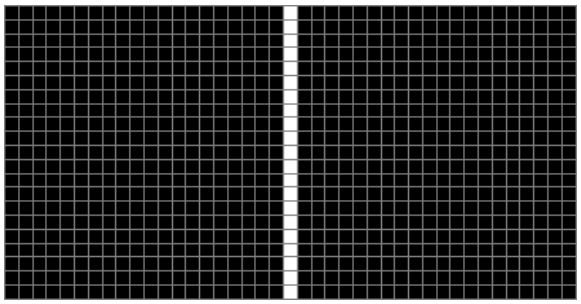
Rule #255



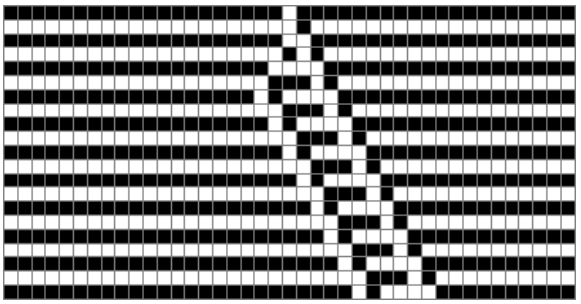
Rule #7



Rule #127



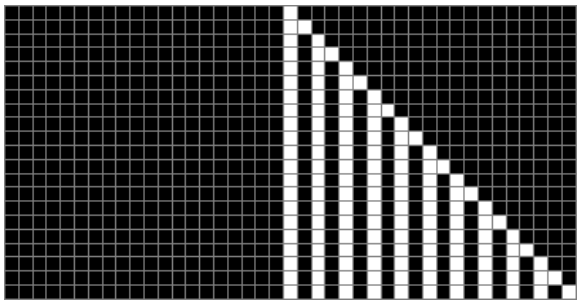
Rule #4



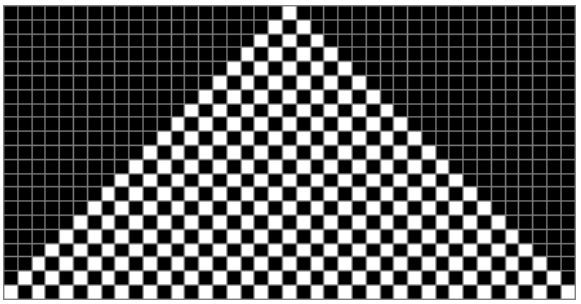
Rule #103

In these cases, the patterns that emerge from a predictable rule do not really grow. However, repetitive patterns, like the ones found in terraces #28, #50, #109, and even #220, can keep growing forever. These terraces can emulate certain number sequences, if you pretend the states of a lantern are bits in a number's binary representation. #220, for example, features the Mersenne numbers (1, 3, 7, 15, 31, 63, …), numbers that are one less than a power of two, and would have a closed form expression of $f(t) = 2 \cdot 2^t - 1$, where $t$ is the number of days passed in the kingdom. Also, rule #28 features the Jacobsthal numbers (1, 3, 5, 11, 21,

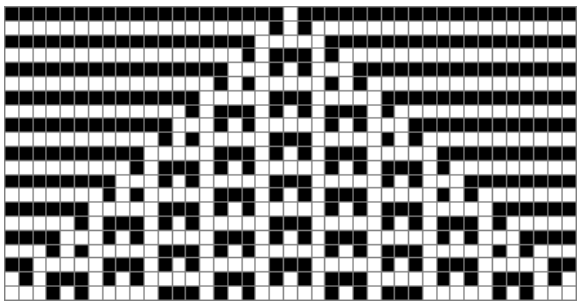
43, …), numbers that come from a sequence similar to the Fibonacci sequence, although you add twice the number that comes before the current number to it.



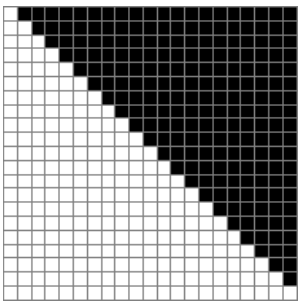
Rule #28



Rule #50

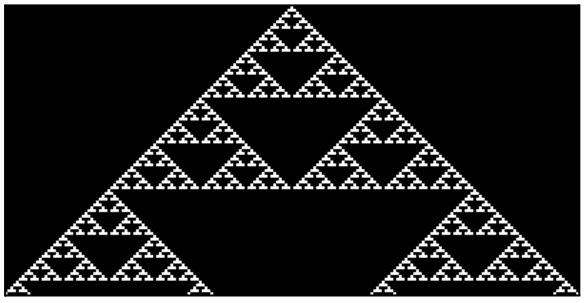


Rule #109

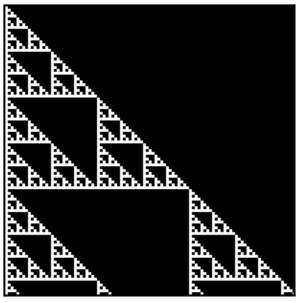


Rule #220

Most of such patterns are nested lantern patterns, like in terraces #22 and #60. #60 also has a unique mathematical property, this time something to do with Pascal's triangle[6]. Hypothetically, after every day, if the terrace would emulate the rows of the triangle such that a lantern is lit once a number is odd, the fractal pattern equivalent to #60 would emerge.



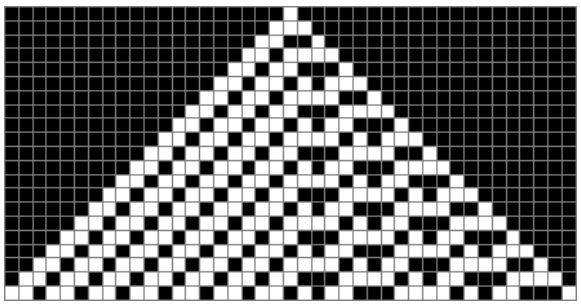
Rule #22



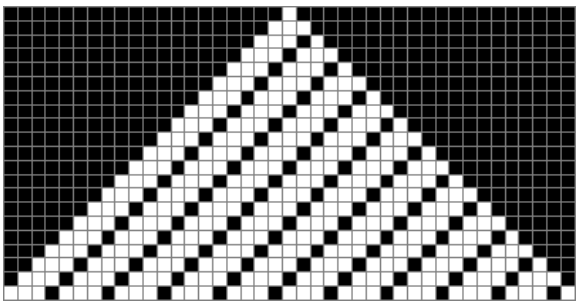
Rule #60

Patterns that are nested and repeated may also identify numbers in the way the lights flicker in the middle. Consider #62, #190, and #129, which enumerates the multiples of 3, multiples of 4, and the powers of two respectively.

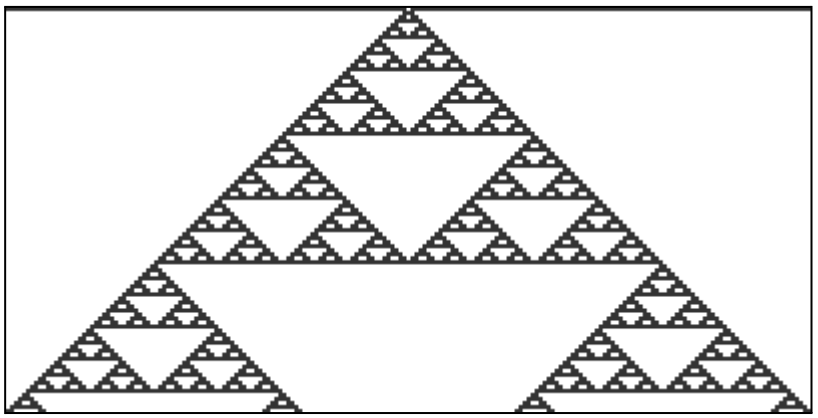
[6] A triangular array of numbers where each number is the sum of the two numbers above it.



Rule #62

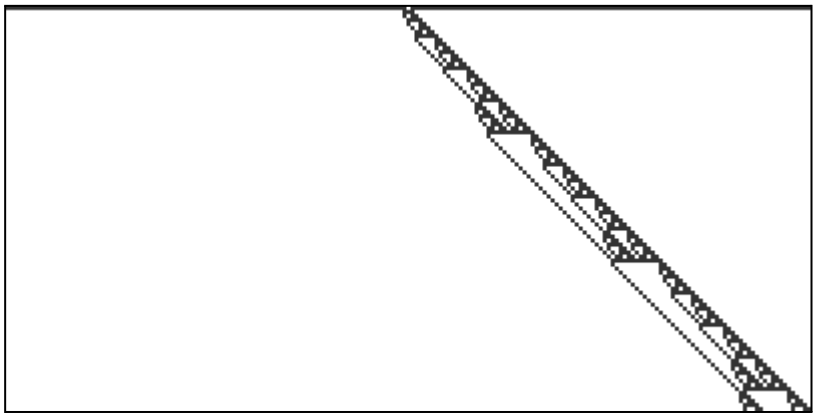


Rule #120



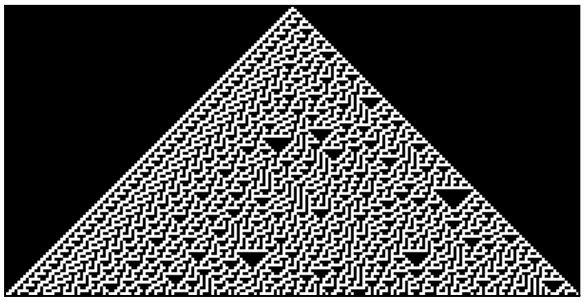
Rule #129

One of the more interesting “mathematical” terraces is #225, which may neither seem repetitive nor nested in nature. However, the pattern, like the moving predictable patterns we saw before, shifts right one house each day. When the pattern is shifted, you will realize that the pattern grows at a rate of $\sqrt{t}$, where t is the number of days.

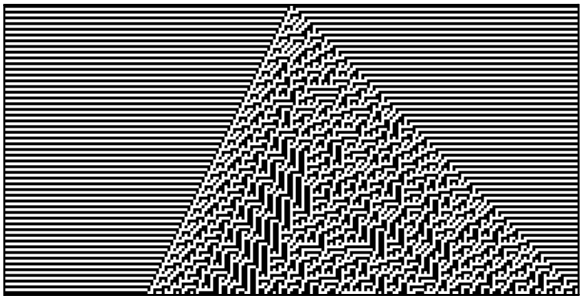


Rule #225

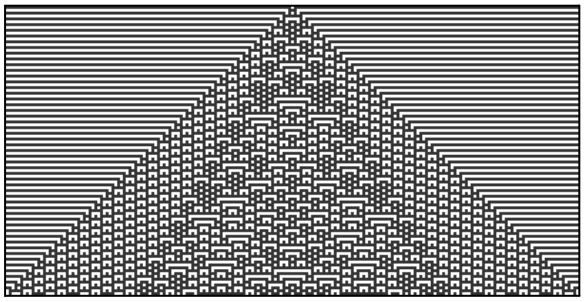
Besides the common theme of repetition and nesting in the kingdom of Elemento, there are also the types of terraces that have patterns with fundamentally random features, like #30, #45, and #73, the former of which we have seen. Finally, we end with terraces with a blend of straightforward and irregular patterns, like rule #110, as we have seen before.



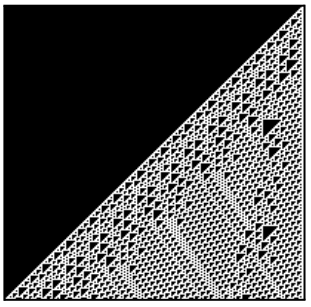
Rule #30



Rule #45



Rule #45

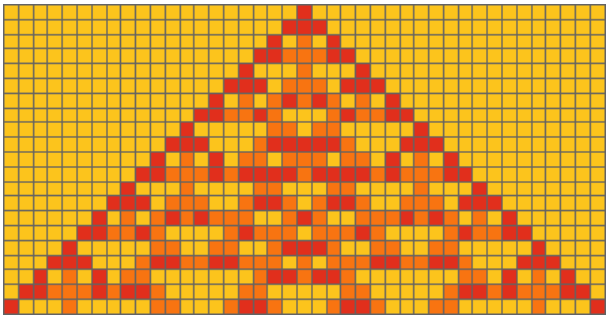


Rule #110

**Further complicating the rules**

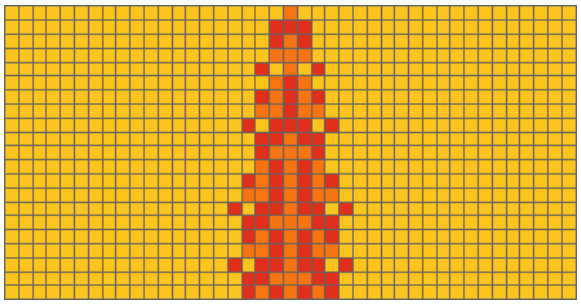
We have continuously shown that even from a simple set of arbitrary laws and a simple initial condition, the terraces of Elemento still flicker in diverse ways that can either be very predictable or erratic and can even hide mathematical computations in binary. But, how about when the rules themselves get more complicated? Will the rules induce more complex behavior among the flickering lanterns? Let's visit another kingdom, bordering Elemento, named Totalia.

This time, instead of extinguishing the lantern in Elemento fashion, the residents of Totalia wake up every day to give their lantern's light a different color, either red, orange, or yellow, according to the color of their and their neighbors' lanterns the day before. Given the 3 possible colors from 27 different configurations of lanterns, there would be $3^{27}$ or 7,625,597,484,987 rules for 7,625,597,484,987 terraces; however, their budget is very limited, so they settled instead for $3^{7}$ or 2,187 rules for 2,187 terraces instead, for each one of their flames now have to be based on the average color of them and their neighbor's flames the day before. We can number these rules with the same previous premise, sort the average color values from yellow to red then interpret the resulting colors as a base 3 number instead.

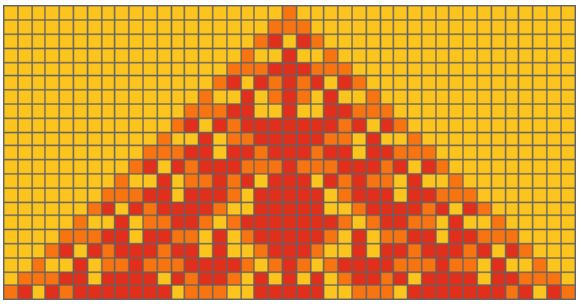


Rule #777

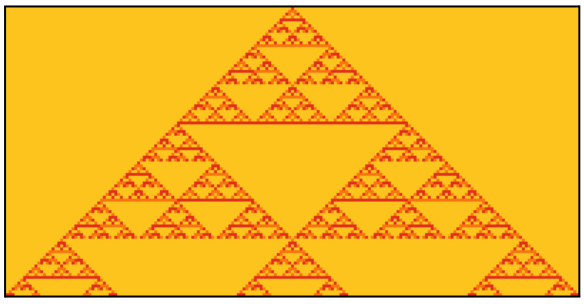
Starting from a single orange flame surrounded by red flames, you would expect for the kingdom of Totalia to have more complex interactions between its lights, as we modified the number of possible states from two to three. However, the complexity of the behavior shown in the kingdom is just as complex and does not add any new features. There are still patterns that repeat either in a fixed size or grow infinitely, and other patterns have nested fractal patterns with some even being featured in Elemento terraces.



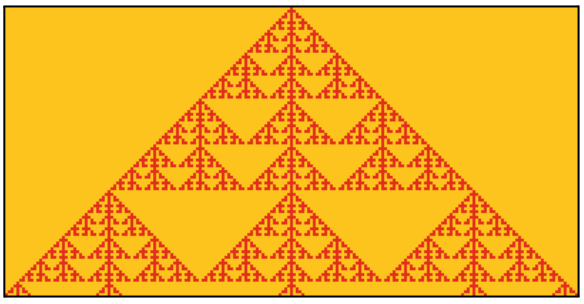
Rule #1167



Rule #1884

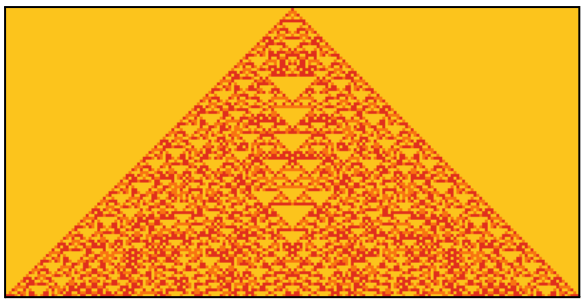


Rule #420

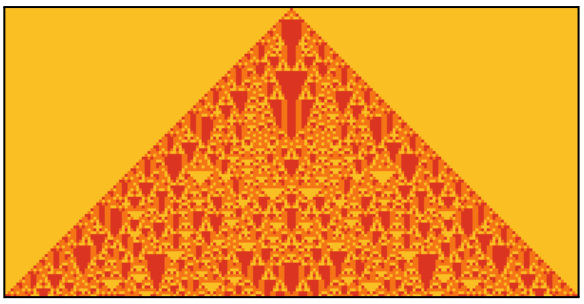


Rule #1749

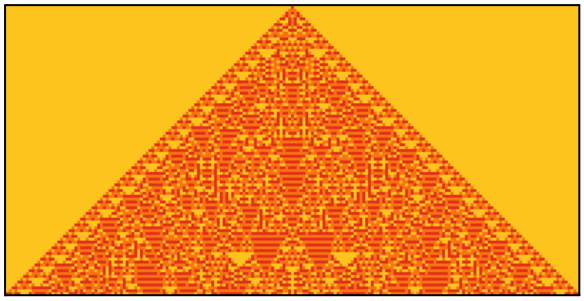
Indeed, some still have patterns with essentially random features and a mix of regular and irregular structures that interact in complex ways. Such features of both types may resolve back into a simple predictable form, but the occurrence of this is hard to predict.



Rule #177



Rule #2040



Rule #1041

Rule #1635

**Conclusion**

All things considered, the flickering patterns of each lantern in Elemento, formally called *elementary cellular automaton*, and the vibrant structures of each flame in Totalia, formally called *totalistic cellular automaton*, prove that a system of rules need not to be complex for the outcome to be complex as well, but rather, complexity emerges when simple rules combine and interact to create patterns that are intricate and unpredictable. Moreover, the consequences of one's emergent behavior do not become more sophisticated with the complexity of one's inherent nature as sophistication comes from the many structures and hidden patterns that develop over time.