Complexity in Simplicity: the Mathematics behind Music.

By Mathias Pettenati

Introduction

What if disorder had a formula? How is chaos created? György Ligeti's piano étude *Désordre* might sound, to the untrained ear, like chaos. Two hands racing across the keyboard, perpetually out of sync. However, beneath the surface of the piece, there lies a carefully crafted structure. This is the same mathematical skeletal system that generates snowflakes, coastlines, and the hidden complexities of the natural world. One rule, repeated forever. This is the heartbeat of fractal geometry and, surprisingly, of *Désordre*. Ligeti strips the music down to the simplest possible instruction, then lets mathematics define the rest. What at first sounds like a monkey slamming the keyboard is a display of mathematical genius through the use of fractals in sound. The Mandelbrot set conjures infinite complexity from a single equation while Ligeti conjures infinite disorder from a single rule. What follows is not a lesson in music, but rather a genius use of mathematics that completely revolutionizes our understanding of composition.

Theory & mathematics

György Ligeti's composition includes many intricately defined layers, much like Mandelbrot's set or Koch's snowflake. The right hand is played on only the white keys, containing four sub-phrases – sub phrases can be thought of as a musical idea that makes up a larger motif. The left hand plays on only black keys, and instead of four sub-phrases, it contains five. This establishes a sense of distortion within the piece as 4 and 5 do not share a common factor, so they will only return to the same position after $4 * 5 = 20$ sub-phrases. On top of this, Ligeti applies a rule to the performance; the right hand loses one eighth note in every phrase while the left hand repeats the same phrase throughout the piece. In other words, the right hand changes what it plays and the left hand does not. This will obviously desynchronize the piece, where the left hand no longer starts its phrase at the same time as the right hand does. The displacement of the notes is highlighted in the image below.

Fig 1: *Khachatryan, Grigor. "Desynchronization of the Left and Right Hand in Désordre by György Ligeti," Jacobs School of Music, 2015.*

The piece progresses and a displacement pattern emerges; the first phrase has no displacement, the second phrase has a displacement of 1 eighth note, the third phrase a displacement of 3 eighth notes, the fourth of 6 eighth notes, and the 5th of 10 eighth notes. Here, the number of displaced eighth notes forms an arithmetic sequence and you may notice that they are also the triangular numbers.

Both the concept of fractals and *Désordre* incorporate self-similarity, when a structure appears roughly the same across a number of scales. If we zoom in to one specific phrase, we would find that the left and right hand begin together, and by the end of the phrase, they are displaced by one eighth note. As mentioned previously, the displacement after each cycle forms the triangular numbers where phrases in the left hand become further and further displaced from phrases in the right hand. Zooming out to the wider body of the piece reveals the same pattern as the two previous ones; a melodic shape which begins synchronized and ends entirely dislocated. This is representative of fractals because zooming into a fractal reveals almost-identical or slightly distorted versions of larger-scale shapes.

This three-level self-similarity mirrors the recursive structure of a fractal. In the geometry of fractals, self-similarity comes from the iterations included, meaning that the same rule is applied to its own output forever. In the Mandelbrot set, one equation is iterated forever. In *Désordre*, Ligeti's rule is the removal of one eighth note per phrase, iterated across the whole piece. The displacement values that result (0, 1, 3, 6, 10) are the triangular numbers, given by the formula $T_a = n(n+1)/2$. These are not a coincidence, they are the direct mathematical consequence of adding one more unit of displacement each phrase, the same way fractal iteration adds one more layer of complexity each pass. One rule, producing infinite structure. Overall, both Ligeti and Mandelbrot iterate the same rule over and over, generating identical structures across every scale.

Another aspect of fractals that can be observed in music is that of the 1/f dimension. In mathematics, the first dimension consists of a line. Doubling the length of a line leaves you with 2 times as much line, and if you square this, you get four sides (or a square). This is the second dimension. If you cube 2, you arrive at the third dimension and result in a cube. This is important because in both Ligeti's music and in Koch's snowflakes, an in-between dimension is introduced; the 1/f dimension. In music, the first dimension is linear and only one note is played. On a graph, it may look like this:

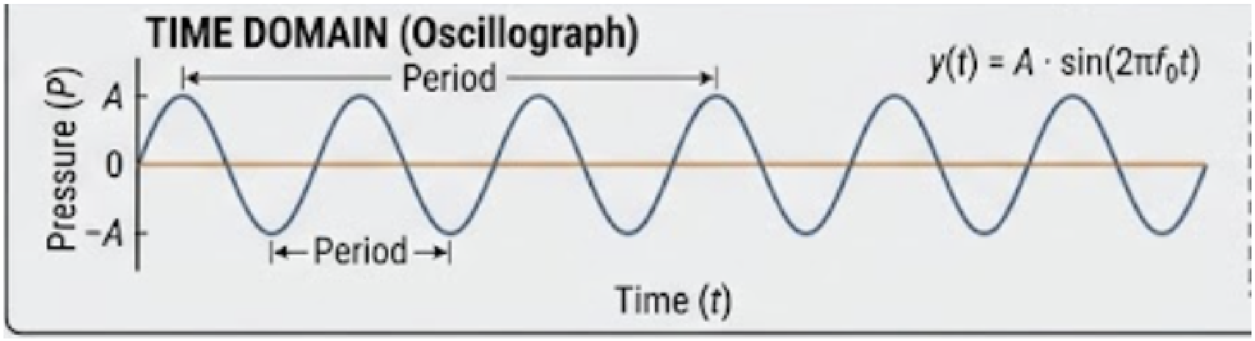

Fig 2: *Gemini. Prompted on "sound waves at the 1st dimension." 2026.*

The second dimension is conversely very random, containing completely independent events. This is the way it appears on a graph:

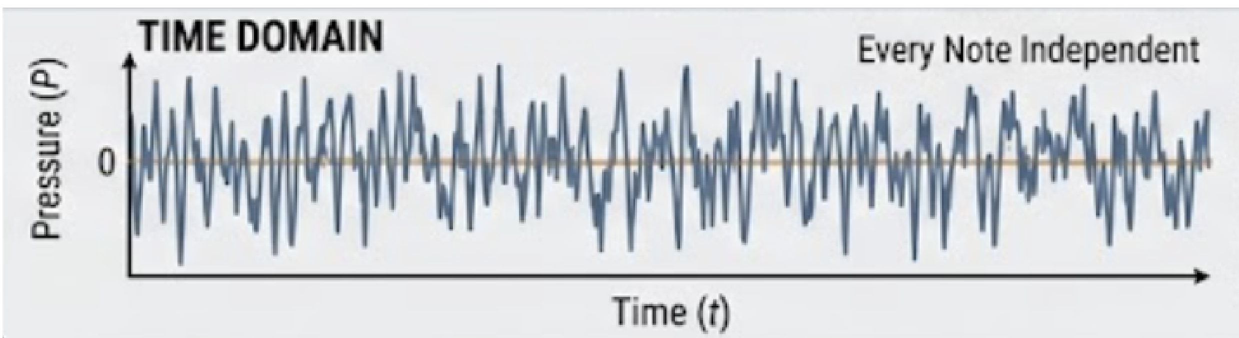

Fig 3: *Gemini. Prompted on "sound waves at the 2nd dimension." 2026.*

The in-between 1/f dimension contains self-similarity on a number of scales because it is the fractal dimension of music. Here are 1/f dimensions depicted on a graph:

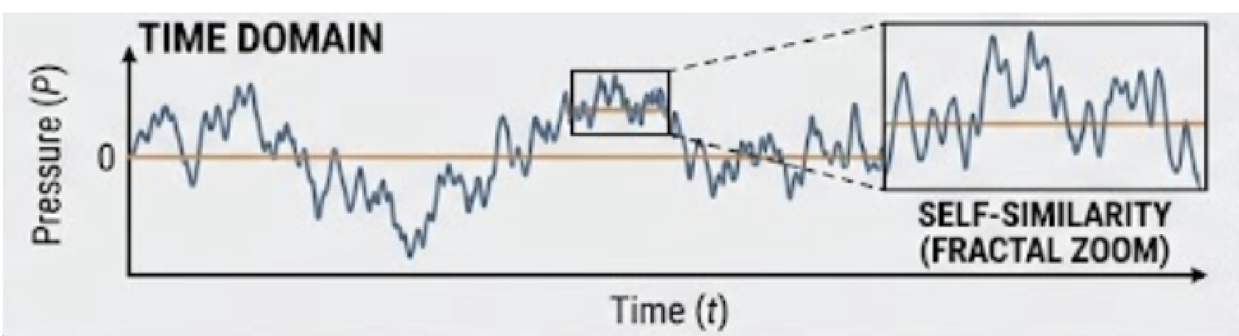

Fig 4: *Gemini. Prompted on "sound waves at the 1/f fractal dimension." 2026.*

These graphs share the same scaling property as that of the Koch curve. For reference, here is the Koch curve:

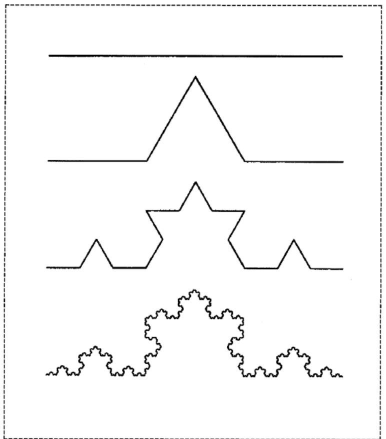

Fig 5: *Steinitz, Richard. "Koch's Curve," The Musical Times, 1996.*

The Koch snowflake has a Hausdorff dimension of approximately 1.26. This is modeled by the equation $d = log(4) / log(3) \approx 1.26$ where each iteration replaces 1 segment with 4 new segments, each 3 times smaller than the original. Like the 1/f dimension in music, the Koch dimension is between the first and second dimension.

To understand why the 1/f dimension sits between 1 and 2, it helps to see how dimension is calculated for fractals. For any self-similar shape made of N copies of itself and being scaled down by a factor of *r*, the dimension *d* is:

D = log(N)/log(1/r)

Two examples make this concrete:

- **Lines**: Take a line and cut it in half. You get 2 smaller lines. Each line is half the size of the original, meaning you plug in $N = 2$, $r = ½$, and then you get log(2)/log(2) = 1. That confirms that a line is 1-dimensional.
- **Squares:** A square is also a good example to demonstrate dimensional properties. Take a square and cut each side in half, giving you 4 smaller squares each being half the size. Plug in $N = 4$, $r = ½$, and you get log(4)/log(2) = 2. This confirms again that a square is 2-dimensional.

Going back to Koch's Curve is where it gets interesting. Each segment of the Koch curve is replaced by 4 new segments, but each one is only ⅓ the size of the original. Plug in $N = 4$, $r = ⅓$ and you get log(4)/log(3) ≈ 1.26. This clearly not being a whole number implies that the Koch curve is more complex than a line but never fills a surface, so mathematically it sits in between. The reason the Koch curve returns a non-integer dimension is because N and 1/*r* are mismatched. Whilst the curve produces 4 copies, each copy is scaled by one-third, not one-fourth. In the line, 2 = 2^1. In the square, 4 = 2^2. The Koch curve breaks this: $4 \neq 3^d$ for any integer number *d*, not real number, as a solution would be log(4)/log(3). Therefore, the dimension that satisfies the equation is fractional. In fact, it aligns with the formal definition for a fractal: A shape whose dimension is not a whole number, in between a line and a surface.

Going back to the application of music, the 1/f dimension in music occupies precisely this same space. In a 1/f signal, power is inversely proportional to frequency, meaning as the frequency increases, the power decreases. This means slow, large-scale patterns have a greater presence in the signal than rapid, small scale fluctuations. Low frequencies involve slow and large-scale fluctuations (as per **Fig 2**), whereas high frequency involves fast and smaller scale fluctuations (as per **Fig 3**). White noise, a constant sound that combines all audible frequencies at equal intensity, sits at one extreme, where all frequencies (the rate at which a sound wave repeats per second) are equal and have moments independent of each other. A pure tone sits at the other, where one frequency is dominant. Ligeti's masterpiece sits in the dimension in between, where one repeated rule provides the piece with enough order and enough chaos to produce a work perpetually out of sync.

Conclusion

Let us return to the sound of *Désordre*: the same desynchronized notes, the same rushing melodies, and the same mounting disorder. You now understand the rule: an iterated cycle of one eighth note dropping every phrase. Ligeti did not compose disorder. Rather, he composed a musical instruction that, when understood, permits you to understand the method to Ligeti's 'madness'. Similarly, the Koch snowflake and the Mandelbrot set were not constructed purely on their own but were the products of one iterated rule or formula. Through *Désordre* and his many musical works, Ligeti reveals that beneath absolute disorder there is one simple formula. And within it lies the beauty of fractal geometry.

Works Cited

Khachatryan, Grigor. *GYÖRGY LIGETI'S DÉSORDRE: MUSICAL CHAOS ACHIEVED with ORDER*. 2015.

Steinitz, Richard. "The Dynamics of Disorder." *The Musical Times*, vol. 137, no. 1839, May 1996, p. 7, https://doi.org/10.2307/1003934. Accessed 12 Apr. 2026.