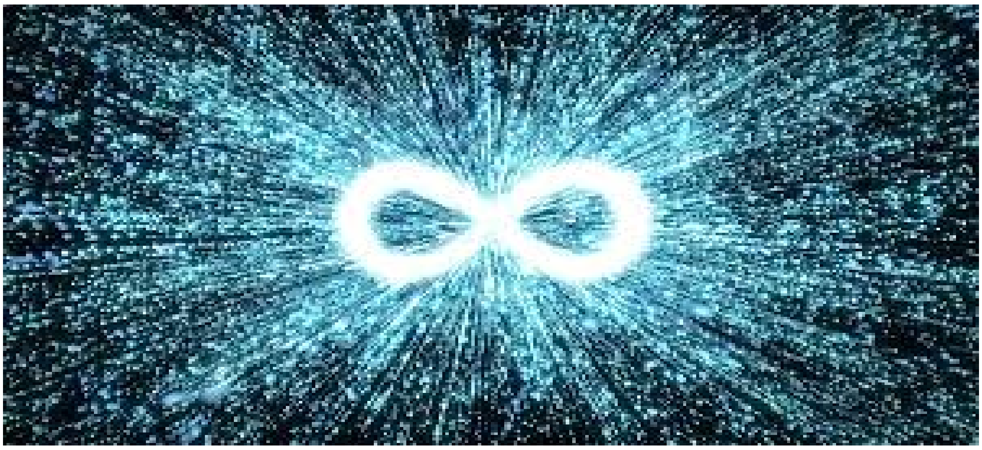
Beyond Endless:The Mathematics of Larger Infinities



1.Introduction:

The concept of infinity is often seen as a limitless, singular concept. However this seemingly simplistic view of infinity masks a more counterintuitive idea at the heart of infinity: not all infinities are the same size. While this may seem counterintuitive to have one endless list bigger than another, Georg Cantor's work in the 1800s helped show that infinities can differ and that these different infinities can be compared and even measured.

Through this essay I will explore how earlier mathematicians were able to break that barrier and make sense of infinity not just by glossing over it like most others had done before, but rather treating it as something that should be examined more rigorously, observed and measured through their various cardinalities. By comparing different sets of infinity we can start to see how some sets, e.g the positive integers, seem to be "countable" while others, such as the real numbers, are "uncountable".

This is evident as we look more closely at the work of Georg Cantor, namely his Diagonal argument, which seeks to show that no complete list can be assembled for certain sets of infinity, most infamously the real numbers. His work shows some infinities as larger than others, raising deep and important questions about the very nature of infinity, for instance "If infinity is not singular and infinite then what is it?" furthermore raising the question "How far does the hierarchy of infinity extend?"

2.What is Infinity?

Imagine gazing up at the night sky, stars stretching as far as you can see, now imagine a concept far greater than the vastness of our cosmos, an endless expanse without limit, this is infinity. The symbol for this was first introduced by the mathematician John Wallis in 1655 and it's representative of infinity as it has no beginning or end, just a never-ending loop. Contrary to popular belief, infinity isn't just an individual concept but rather has multiple variants. There is potential infinity which is an endless process such as counting higher like the infinite list of integers and there is actual infinity which is a completed infinite set like the full list of natural numbers. To measure and compare infinities mathematicians utilise clever proofs such as Cantor's Diagonal argument and cardinalities to show that some infinite sets are larger than others e.g the set of real numbers is larger than the set of natural numbers. These initial proofs create an infinity ladder, one in which each step is a different infinite set and every step up corresponds to a larger cardinality. The concept of infinity is key in mathematics, however it also has applications elsewhere such as physics and philosophy. While its applications in physics are only theoretical and speculative as some people believe the universe is infinite or that matter is infinitely divisible, its application in philosophy is more of a core concept when talking about God and the proofs surrounding it. It is even a quality God inhabits himself.

Infinite set of natural numbers:

$$\mathbb{N}=\{1,2,3,\ldots\}$$

3.Different types of infinity:

Countable infinity:

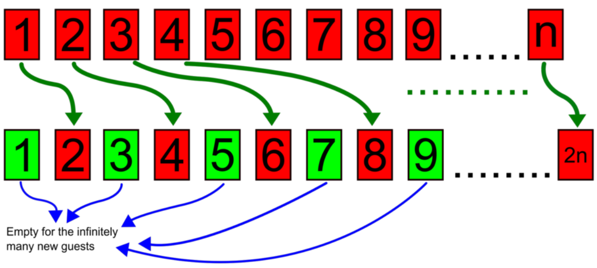
A countably infinite set is any set which can be put in a one-to-one correspondence with $\mathbb{N}$. Once set X is proven countable, any other set which can be put into one-to-one correspondence with X is also countable. Countably infinite sets have cardinal numbers aleph-0 or aleph-null($\aleph_0$).

A good example of another countably infinite set is the set of $\{1/n : n \in \mathbb{N}\}$. You can map $n \mapsto 1/n$ simply by putting 1 over the corresponding term in $\mathbb{N}$.

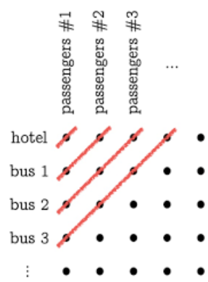
$\mathbb{N}=\{1,2,3,\ldots\}$	$\{1/n:n\in\mathbb{N}\}$
1	1/1
2	½
3	⅓

These sets are interesting to compare because while they are both countable infinities and seem very similar, their behaviours are wildly different. The set of natural numbers will tend to infinity due to it being unbounded and lacking a largest element, on the other hand the set of $\{1/n : n \in \mathbb{N}\}$ tends to 0 as $n \to \infty$.

The Hilbert Hotel problem helps show the application of countable infinities and goes as follows:
Imagine a hotel that has an infinite number of rooms (1,2,3,...) whereby only 1 person is allowed in each room and every room is full. It may seem impossible to add any more guests, however, you are always able to since we are only dealing with a countable infinity.
The Hilbert Hotel presents several problems:
The first problem is a finite number of new guests. With one additional guest, the hotel can accommodate them if every current guest moves down 1 room, moving every guest from their current room, n, to their new room, n+1. The infinite hotel has no final room, so every guest has a room to go to. After this, room 1 is empty and the new guest can be moved into that room. By repeating this procedure, it is possible to make room for any finite number of new guests. To generalise this, when k guests seek a room, the hotel can apply the same procedure and move every guest from room n to room n+ k.
The second problem is a coach with an infinite number of new guests, however you can overcome this as the new guests are still countably infinite so you can tell every current guest to move from their room n to their new room 2n, this would leave all odd numbered rooms vacant and we know that there is a countably infinite number of odd numbers so all guests can slot in perfectly.



The third problem you would encounter is when infinitely many coaches with infinitely many new guests arrive. There are multiple ways you can deal with this,
One method is prime factorisation:
You start by assigning each coach a number, c, and each seat a number, n, each person is now put into a room using the expression $(2^n)(3^c)$. Each positive integer has a unique set of prime factors meaning no rooms will overlap and everyone will have a room.
An alternative method is the Interleaving method:
For this method you start by making a table where each row denotes a bus and each column represents each room number occupied in the hotel, creating a sequence that lists every pair (Bus,Seat). This means each person has a unique identifier which is a combination of their bus and seat number. You then draw a string that zigzags back and forth over every identifier. Then pull the string taut and this will make a single, infinite, straight line which you can then easily match up to each room in the hotel accommodating all of the guests.



All of these problems can be overcome because we are dealing with the smallest infinite cardinality, $\aleph_0$. This flexibility is a defining feature of countable infinities. However, this breaks down when we consider a 4th problem: dealing with an uncountably infinite number of new guests. This would be impossible because it is impossible to map or reorder uncountable guests into a countable set of rooms without leaving infinitely many guests without rooms. What makes this result so surprising is that it shows infinity behaving unexpectedly as a “full” hotel can accommodate infinitely more guests.

Uncountable infinity:

Uncountably infinite means that the set cannot be put in one-to-one correspondence with the infinite set of natural numbers. These uncountably infinite sets therefore have a cardinal number greater than $\aleph_0$, however they can have different cardinalities. For example the set of real numbers ($\mathbb{R}$) is uncountably infinite, and its cardinality is denoted by $2^{\aleph_0}$ (the cardinality of the continuum).

The Power Set of real numbers, $P(\mathbb{R})$ is the set of all subsets of the real numbers. According to Cantor's theorem, the cardinality of $P(\mathbb{R})$, denoted as $2^{(2^{\aleph_0})}$, is strictly greater than the cardinality of $\mathbb{R}$.

One example of an uncountably infinite set is the set of all real numbers:

$$\{x : x \in \mathbb{R}\}$$

We can see that this set is uncountably infinite by using Georg Cantor’s Diagonal argument.

$$\begin{array}{ll}
s_1 & = 0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\ldots \\
s_2 & = 1\,1\,1\,1\,1\,1\,1\,1\,1\,1\,1\ldots \\
s_3 & = 0\,1\,0\,1\,0\,1\,0\,1\,0\,1\,0\ldots \\
s_4 & = 1\,0\,1\,0\,1\,0\,1\,0\,1\,0\,1\ldots \\
s_5 & = 1\,1\,0\,1\,0\,1\,1\,0\,1\,0\,1\ldots \\
s_6 & = 0\,0\,1\,1\,0\,1\,1\,0\,1\,1\,0\ldots \\
s_7 & = 1\,0\,0\,0\,1\,0\,0\,0\,1\,0\,0\ldots \\
s_8 & = 0\,0\,1\,1\,0\,0\,1\,1\,0\,0\,1\ldots \\
s_9 & = 1\,1\,0\,0\,1\,1\,0\,0\,1\,1\,0\ldots \\
s_{10} & = 1\,1\,0\,1\,1\,1\,0\,0\,1\,0\,1\ldots \\
s_{11} & = 1\,1\,0\,1\,0\,1\,0\,0\,1\,0\,0\ldots \\
\vdots & \quad \vdots\,\vdots\,\vdots\,\vdots\,\vdots\,\vdots\,\vdots\,\vdots\,\vdots\,\vdots\,\vdots\ \ddots \\
\\
s & = 1\,0\,1\,1\,1\,0\,1\,0\,0\,1\,1\ldots
\end{array}$$

Firstly, you assign each natural number to a real number and repeat this for every natural number. Cantor's aim with the Diagonal argument is to create a real number that appears nowhere in this infinite list of real numbers. He does this by looking at the first digit of the first number in the list and ensures the first digit of his new number differs, then he looks at the second digit of the second number and ensures it differs in his new number and he continues this an infinite number of times guaranteeing that his new number will be different from every number in the list as the first digit of his number is different to the first digit of the first number, his second digit is different from the second digit of the second number and so on and so forth. This shows that we can always create a new number that doesn't appear anywhere in the list, showing that no list can contain every real number and therefore the set is uncountable, as shown in Cantor’s Diagonal argument (1891). This argument highlights the different types of infinity showing that infinity cannot be categorised into any single “size”. This leads to the Continuum hypothesis: whether an infinity exists between the real and natural numbers. This hypothesis was proposed by Cantor and states that $2^{\aleph_0}=\aleph_1$, although this can neither be proven nor disproven within standard axioms, as it is independent of ZFC.

4.Why does it matter?

Firstly, Cantor's work created set theory, which is the foundation for important areas of advanced Mathematics, specifically:
1.Calculus - It provides the fundamental language by defining the real number system and outlining limits and continuity e.g epsilon-delta definition of limits, which underlies derivatives and integrals, relies heavily on set notation and understanding properties of sets of real numbers.
2.Analysis - Set theory acts as the foundational language of mathematical analysis by providing the framework of ZFC(the standard foundational axiomatic system for mathematics) to define fundamental objects.
3.Topology - It creates foundations by defining topological spaces as specific collections of subsets acting on a set of points and helps with continuity

Secondly, it has led to advancements in computing as ideas about countability allow us to know what problems computers can solve and the limits of algorithms. One example is the foundation it gives to undecidability, namely Cantor's Diagonal argument and his proofs on the difference between countable and uncountable infinities.

5.Conclusion:

Infinity, a concept often seen as a single idea, when examined more closely we start to unveil a rich, seemingly infinite hierarchy. By comparing countable and uncountable sets we realise the unorthodox, counterintuitive truth that some infinities are larger than others. This truth challenges intuition and alters our understanding of size as a concept. Ultimately, infinity isn't a boundary of mathematics but rather a path to deeper and more profound questions about the nature of reality.

6.Bibliography:

Wikipedia Contributors (2025). *Actual and potential infinity*. Wikipedia.

Weisstein, E.W. (n.d.). *Countably Infinite*. [online] mathworld.wolfram.com. Available at: https://mathworld.wolfram.com/CountablyInfinite.html.

Veritasium (2021). *How An Infinite Hotel Ran Out Of Room*. [online] www.youtube.com. Available at: https://www.youtube.com/watch?v=OxGsU8oIWjY.

Wikipedia Contributors (2025). *Cantor's diagonal argument*. Available at: https://en.wikipedia.org/wiki/Cantor%27s_diagonal_argument

Weisstein, E.W. (n.d.). *Cantor Diagonal Method*. MathWorld—A Wolfram Web Resource. Available at: https://mathworld.wolfram.com/CantorDiagonalMethod.html

Doing Maths (2021). *Hilbert's Hotel Paradox - A Stay at the Infinity Hotel - A Mathematics Story*. *YouTube*. Available at: https://www.youtube.com/watch?v=NHtZqbR8CSw [Accessed 15 Feb. 2022].