Countable and Uncountable Infinities

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1 Introduction

You find yourself in a ball pit surrounded by infinitely many balls and have been tasked with counting each one without missing any or counting any twice. Despite being a rather mean task to give someone, it *is* mathematically possible to do so. Now imagine the same task, but this time with every real number between zero and one. Suddenly — and provably — it is not.

Two infinite sets — yet only one can be counted. Why?

At first, this seems strange, since the Oxford English Dictionary defines infinity as "An indefinitely large number or amount," suggesting that all infinities behave in the same way. We will soon see what David Hilbert and Georg Cantor had to say about this...

2 What do we mean by "Size"?

Since the dawn of mathematics, we have been counting things. Five students, three snakes, one unfortunate incident involving both. These collections of objects are all countable and comparable. Consider the two sets below.

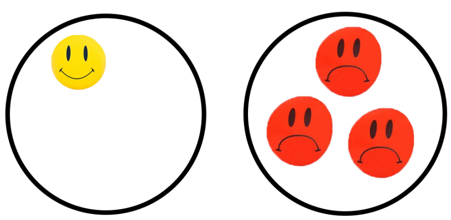

Figure 1: Happy and Sad Faces

As we can see, there are more sad faces than happy faces, although perhaps that will change by the end of this essay. Since there are more sad faces, it is reasonable to conclude that the second set is larger.

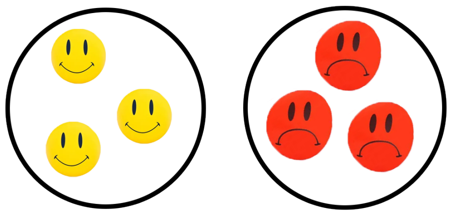

Figure 2: More Happy and Sad Faces

Now, in the sets above, there is an equal number of smiley faces and sad faces. Mathematicians refer to this as a *bijection* – a one-to-one correspondence between two sets, where each element in one set is paired with exactly one element in the other, with nothing left unpaired. From this we can draw the conclusion that the sets have the same size.

This idea also lends itself naturally to collections of numbers.

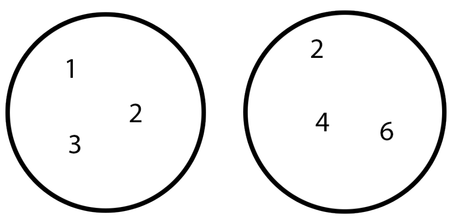

Figure 3: Small finite number sets

Although the elements are different, the *number* of elements in each set is the same. We can still pair each element in one set with exactly one element in the other set, so the sets have the same size. This shows that equal size does not require a particular relationship between the elements themselves.

So far, everything matches our intuition. But what happens when, much like the infinite ball pit I have trapped you in, these sets are infinite? This idea of pairing elements will be our key tool for comparing infinite sets.

3 Countable Infinities

Below are two sets: the natural numbers and the even numbers.

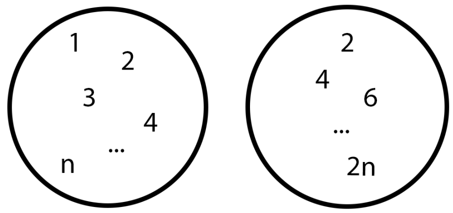

Figure 4: Infinite number sets

If we were comparing the number of elements in each set, the size of the natural numbers would be greater than the size of the even numbers, right? Wrong. This delightfully counterintuitive result is the cornerstone of countable infinities. At first, it might seem that there should be twice as many natural numbers as even numbers, yet we can pair every natural number n with the even number $2n$, with nothing left unpaired. By our definition of bijection, they must be the same size. Because of this, we say that they have the same cardinality.

If a set can be listed like this, element by element in a one-to-one correspondence with the natural numbers, we call it *countable*.

At this point, it is tempting to conclude that all infinite sets must be the same size. After all, even when one set appears to be larger, we can still construct a bijection between them. But is this always the case? Could there exist infinite sets that are genuinely larger than others, even in this new sense of size?

To better understand the strange nature of countable infinities, we must now turn to one of the most famous thought experiments in all of mathematics.

4 Hilbert's Hotel

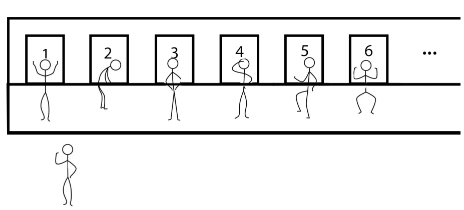

Figure 5: Hilbert's Hotel

Imagine a hotel consisting of a corridor with infinitely many rooms, each numbered one, two, three, four, and so on, without end. Suppose that infinitely many guests arrive, looking for a room. Setting aside the moral challenge faced when assigning someone a room infinitely far down the corridor, we pair each guest with a room, and so the hotel is full.

But what happens when one more guest arrives?

Fortunately, as the manager of such a hotel, you are familiar with the peculiar properties of infinity. You order each guest in room n to move to room $n + 1$, thereby freeing room one for the new arrival. In this way, each guest is paired with exactly one room and each room is paired with exactly one guest. The natural numbers and the natural numbers plus one extra guest have the same cardinality — infinity, evidently, has room to spare.

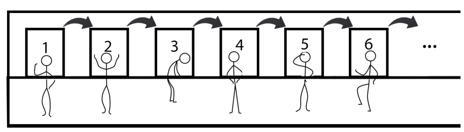

Figure 6: Additional Guest

Even more remarkably, this hotel can accommodate not only one additional guest, but infinitely many. Consider the sets that were compared earlier: natural numbers and even numbers. By instructing your guests to move from room n to room $2n$, all the odd rooms would be left vacant. These rooms themselves are infinite in number, and so each new guest can be assigned to one of them.

This process mirrors the bijection between the natural numbers and the even numbers, and demonstrates how the natural numbers, even numbers and odd numbers all share the same cardinality, denoted by $\aleph_0$.

So far, the infinite sets we have looked at would all be considered *countable* — yet not all infinite sets are so accommodating.

5 Uncountable Infinities

Now, hopefully you haven't forgotten about your earlier ball pit predicament, so let us make life a little harder. Suddenly, the plastic balls around you vanish, each one replaced by a real number between 0 and 1. Endless decimal expansions levitate around you — floating point numbers, if you will — each representing a real number.

You reach out and select one, say 0.371041... But what's the *next* one? There's no such thing. No matter how close two real numbers are to each other, there is always a number between them.

Intuitively, this makes sense. But we mathematicians don't accept intuition as proof. To show rigorously that the real numbers aren't a countable set we turn to one of the most elegant arguments in mathematics: Cantor's Diagonal Argument.

6 Cantor's Diagonal Argument

Before we proceed, we must first clarify what we are trying to prove. As suggested by the ball pit analogy, where there is no such thing as a *next* real number, it seems impossible to list all real numbers in a single sequence. In other words, the real numbers cannot be put into a one-to-one correspondence with the natural numbers.

To prove this, we must first assume the opposite: that it is possible to count the real numbers. This assumption will ultimately lead us to a contradiction.

Assume that the real numbers between 0 and 1 *are* countable. Doing so allows us to list them all, paired with the natural numbers:

$$r_1 = 0.371041...$$

$$r_2 = 0.561723...$$

$$r_3 = 0.875854...$$

$$r_4 = 0.141592...$$

$$...$$

and so on, forever.

We can lay these out as their decimal expansions following 0, like so:

	d1	*d2*	*d3*	*d4*	...
$r_1 = 0.$	3	7	1	0	...
$r_2 = 0.$	5	6	1	7	...
$r_3 = 0.$	8	7	5	8	...
$r_4 = 0.$	1	4	1	5	...
...					

Figure 7: Cantor's Diagonal Argument

Assuming this list contains all the real numbers between 0 and 1, we can prove that there must be at least one number that has not yet been included. We will do this by constructing a new number that differs from every number in the list by at least one decimal place.

Now consider the digits along the leading diagonal — the first digit of r_1, the second of r_2, the third of r_3, and so on — giving us 3, 6, 5, 5... To construct a new number, d, that isn't already on the list, we change each of these digits (avoiding representations that terminate in repeating 9s). We can define this number d by choosing its nth decimal digit as follows: if the nth digit of r_n is not 3, let it be 3; otherwise, let it be 4.

Now, ask yourself: is d on the list?

1. It can't be r_1 because it differs in the first decimal place.
2. It can't be r_2 because it differs in the second decimal place.
3. It can't be r_n because it differs in the nth decimal place.

So d is *nowhere* on the list, despite clearly being a real number between 0 and 1. Even if we include d anywhere in the list, we are able to construct another number that should appear on the list but does not, contradicting our assumption that the list contained all real numbers between 0 and 1. This contradiction shows that the real numbers cannot be listed in a one-to-one correspondence with the natural numbers. Therefore, the set of real numbers is uncountable.

Moreover, this shows that the set of real numbers has a strictly greater cardinality than the natural numbers. Through this argument, Cantor demonstrated that there exists an infinity beyond $\aleph_0$.

The cardinality of the reals is called $\mathfrak{c}$. Whether this equals $\aleph_1$ — the very next infinity after $\aleph_0$ — is a question with a remarkable and unexpected answer.

7 The Continuum Hypothesis

So far, we have two infinities: $\aleph_0$ (the natural numbers), and $\mathfrak{c}$ (the reals). Using Cantor's diagonal argument we have proved that $\mathfrak{c}$ is strictly greater than $\aleph_0$. Following this realisation, Cantor showed that for any set, the set of its subsets is always strictly larger, generating an infinite hierarchy of infinities:

$$\aleph_0 < \aleph_1 < \aleph_2 < \aleph_3 < \aleph_4 < \dots$$

Cantor proposed what seemed to be a very simple question: "Is there a set whose size is *between* $\aleph_0$ and $\mathfrak{c}$?" Essentially, the continuum hypothesis states that no such set exists.

In 1940, Kurt Gödel showed that the continuum hypothesis is *unable to be disproved* from the standard axioms — rules that we accept as true — of mathematics. Problem solved, right? Unfortunately this problem is not so simple, as, in 1963, mathematician Paul Cohen showed that it *cannot be proved*, either. Together, this highlights that the continuum hypothesis is independent of the axioms of set theory as we know it.

In the case of the continuum hypothesis, you can assume it's true or false and, in either case, obtain a perfectly consistent system of mathematics. It's not that we haven't found an answer yet, but that there *is* no answer provable within standard axioms.

Mathematics is built on axioms, so finding an exception, a hypothesis that is completely undecidable within the rules of mathematics, is truly jaw-dropping.

8 Conclusion

The Oxford English Dictionary told us that infinity is simply "an indefinitely large number or amount." By now, it should be clear that this is a dramatic oversimplification. While Hilbert's Hotel showed us the strange and counter-intuitive properties of $\aleph_0$, Cantor highlighted that there are infinitely many infinities, each strictly larger than the last. If you're not sick of hearing the word infinity, maybe you will find peace in your little ball pit. It seems unlikely you'll finish counting anytime soon.