Counting to Infinity and the Extent of Mathematical Truth

For most of mathematical history, maths was a complete system. It was believed that any statement could be proven true or false given a set of axioms. In the late 1800s, a problem arose in maths, the Continuum Hypothesis, introduced by Georg Cantor in 1878. Cantor was renowned for his work in branch of maths regarding infinities, but he was stumped by the problem. The problem also became the first of David Hilbert's 23 problems proposed in 1900, creator of Hilbert's program, a suggestion of the definitive qualities of maths. However, Kurt Gödel revolutionised our knowledge of maths, including his shocking proof that the Continuum Hypothesis could not be disproved, and his incompleteness theorems, which toppled the idea of a complete maths Hilbert suggested.

Imagine you are managing a hypothetical hotel with an infinite number of rooms; each assigned a natural number. Each room can only accommodate 1 person. On one exceptionally busy day, you find the hotel is at maximum capacity. Now, logically, when someone attempts to check into the hotel, it should be impossible to find a room for them, given there are no empty rooms. But the nature of infinity does not work the same way as numbers. You tell each occupant in a room to move to the room numbered one higher than their current room. This frees up space in room 1 to contain the customer. From this, we can explain that infinity plus any finite number is still infinity.

Now imagine a similar predicament, but rather than 1 person, an infinitely long coach, containing infinite passengers. The same logic to cater a finite number of people no longer works. You can't simply tell a person to "move to the room numbered your room $+\infty$", no doubt there would be plenty of complaints about the distance to their new room. As every natural number can be doubled, you can tell everyone to move to the room numbered their room number times 2. Likewise, for n infinite buses, each hotel guest moves to the room numbered their current room times n+1. This explains why any finite number of infinities is the same as infinity.

Once again, increasing the 'degree' of infinities, the problem of infinite groups of infinite people. However, you are rescued by the work of Euclid, who proved there are an infinite number of prime numbers. Each group (considering the people in the hotel their own group) can be assigned a distinct prime number, and each member of a group a number. Every person can now be assigned a unique room given by:

$$(p_c)^n$$

where p_c is the c^{th} prime number and n is the room number for hotel guests and the seat number for guests on the infinite coaches. Problems beyond 2 levels of infinity are all solvable by extension of the prime powers method.

Suppose another infinite coach, where each person's name is an infinitely long binary string and every permutation exists on the coach. You find that there is no starting point for infinitely long numbers, so you begin listing people one by one. However, this method encounters a flaw.

Consider the number along the diagonal, and invert every digit (0s become 1s, 1, become 0s). You will find that this name does not appear at any point in the list. Whilst you could just add this to the "end of the list", there is nothing stopping the process of making another name that differs from every other name. Thus, you can produce infinitely many new names that do not appear on the list, all of which are not on the list. In set theory, the process of matching elements of 2 sets is called a bijection. In this case, there is not a bijection between the 2 sets and hence one must be larger than the other. Calculating the number of different names, we find there are 2 values at every digit for infinity digits, giving 2^∞ permutations.

Now that we know of 2 different infinities, we need a method to sorting infinities by size. We first assume that the cardinality of the set of all natural numbers is the "smallest" infinity, naming it aleph null, $\aleph_0$, and name the second infinity, 2^∞, as $\aleph_1$.

We can prove that $\aleph_1$ is equal to the cardinality of ℝ, the set of real numbers. First, we introduce power sets, the set containing all subsets of a set. One characteristic of a power set is:

$$|P(A)| = 2^{|A|}$$

We can use this identity to redefine the question to be 'Is there a bijection between P(ℕ) and ℝ'. To do this, let A be a subset of ℕ which can also be described as an element of P(ℕ). For every possible set A, there is a corresponding infinite binary string that represents this set. This binary string is a sequence where if a natural number, n, is in the set A, the nth digit = 1, else 0. This maps each element of P(ℕ) to an infinite binary number. Then, set the binary string to be a binary decimal number in [0, 1]. Hence, there is a bijection between P(ℕ) and the real numbers in the interval [0, 1]. Finally, there is a bijection between real numbers in this interval and all real numbers through functions, for example the function:

$$\tan(\pi(x - 1/2))$$

For $x = 0$ or 1, the first introduced behaviour of infinities shows that $|[0, 1]| = |(0, 1)|$.

We now question whether other infinities exist between these, which became to be known as the Continuum Hypothesis, which states 'there is no set whose cardinality is strictly between that of the integers and real numbers.' The hypothesis became the first of David Hilbert's 23 open problems proposed in 1900. It was proven that it is impossible to disprove i.e. it is impossible to prove a set with a cardinality between the 2 infinities exists, by Kurt Gödel in 1940, and later proven impossible to prove by Paul Cohen in 1963. This leaves the hypothesis somewhat in a paradoxical state. What was proven is that when assuming the Continuum Hypothesis to be true or false, both lead to consistent maths. This means that the Continuum Hypothesis is independent, and cannot be decided from the established axiom system of set theory, Zermelo-Frankael set theory.

In order to prove a statement true, it must be consistent with axioms, trivial statements that do not require proof. In set theory, this axiomatic system which is used to prove hypotheses is Zermelo-Fraenkel set theory and the axiom of choice (ZFC). However, this system, and in fact every mathematical system, was proven by Gödel to be incomplete with his renowned incompleteness theorem. This theorem suggested that for any mathematical system, there exists statements that cannot be proven by the system. His proof followed the following approach:

Consider the statement: 'This statement is false.' Analysing the truth of this statement reveals that it must be false if true and it must be true if false, creating a paradox. This is known as the liar paradox. Gödel intended to solve these self-referential problems mathematically and without the peculiarities of language. Gödel created his own syntax, or his own language to represent operations. Then, he assigned a number to each symbol (the number 0, functions e.g. successor, variables and operators). Then, for a sequence of these symbols that when put together creates some statement, raise the first prime number, 2, to the power of the first symbol's Gödel number, the second prime number by the Gödel number of the second symbol's Gödel number and the nth prime number by the Gödel number of the nth symbol in the sequence. The product of these numbers is then the Gödel number of the statement. From the fundamental theorem of arithmetic, each possible Gödel number corresponds to only one statement, and every statement has one Gödel number. Now, imagine the statement 'there is no statement with Gödel number G' with a Gödel number, G. In essence, it states that 'this statement cannot be proven' Assume the statement can be proved, then the statement with Gödel number G, itself, is true or false. If true, then the statement is false whereas if false, the statement must be true. In both conditions, the statement is self-contradictory, so the system that defines this statement is inconsistent. Assume the statement is unprovable, the provability statement agrees with what it states and hence the statement is true, which again contradicts itself. What Gödel proved is that in any system, there exists statements that are incomplete, meaning that they cannot be proven or disproved.

Hilbert believed mathematics was undoubtedly 4 things: complete, consistent, conservative and decidable. Gödel's first incompleteness theorem proved that maths was incomplete, there are statements that cannot be proven or disproven, one of which is the Continuum Hypothesis. Gödel also proved in his second incompleteness theorem that for a mathematical system, it is not possible to find the consistency of itself.

Thirdly, conservativity of maths refers to the ability that if a proof can be made from a stronger mathematical system, it should be possible to be proven from a simple system. For example, if a statement about natural numbers can be proven using the arithmetic of infinities, then it can be proven from simpler reasoning. This was also disproven by Gödel in his incompleteness theorems. The Continuum Hypothesis, although not a counterexample of Hilbert’s idea of conservativity, illustrates how statements can only be decided by a system independent of a simpler system, which in this case is ZFC.

Finally, the decidability of maths was disproven by Alan Turing in 1936. Hilbert believed that for a system there is a definite method to determine if a statement can be derived from axioms. Turing used this to develop the Turing machine, the basis for modern computers. The concept of the machine works by taking in an input tape and a program. The input tape in this case is an infinitely long list of binary digits, and the program determines what the machine should do given its current state and the value of the current bit of the input tape, out of some simple operations such as overwrite the current bit, move the tape 1 digit left or right or halt. Halting is a particular instruction of note in the Turing machine, one that allowed Turing to realise that the problem of determining decidability of a statement was related to determining if a program and input of a Turing machine would lead to halting; this became known as the halting problem.

Turing’s proof starts with a Turing machine, H, that can determine if a program halts. When this machine outputs whether a program halts, imagine another Turing machine that takes this output and does the opposite. Now when feeding this greater system its own combined program code, for H and the computer that inverts the output, the machine now has 2 options: determine that the program halts or doesn’t. Whatever the machine assumes is incorrect, like the liar paradox. This means the initial assumption that such a machine works is false, and no algorithm exists to determine if a program halts, or if a statement can be derived from axioms.

The work of Cantor, Gödel and Turing reshaped maths. Cantor proved that the known ideas of infinity were wrong, showing the existence of infinities larger than others. Gödel showed that the pillars of Hilbert’s idealistic maths were all wrong, providing a stronger understanding of mathematical systems, as well as proved the Continuum Hypothesis could not be disproved, and later Cohen proved it could not be proved. Turing is often named the ‘father of computer science’, the creator of the first model of modern computer science, the Turing machine and the ENIAC, the world’s first programmable electronic computer.