An exploration of continuous symmetry

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1 Introduction: What is symmetry?

In everyday life, symmetry is proportion and balance in an object. In maths, however, symmetry has a more precise definition: it is the invariance of a mathematical object under certain transformations. This clarification of symmetry's meaning may seem pedantic but is important as it allows us to interpret the symmetries of non-geometric objects – ones we normally wouldn't apply the term to. We describe symmetries using group theory, a language that reduces emphasis on physical detail through its use of groups. The abstractness of groups allows us to draw surprising structural comparisons between symmetries of seemingly unrelated objects, systems and even analyse symmetries of physical laws; I will explore some of these in my essay.

2 Formal definition of a group

A group is a set - G - combined with a binary operation - $\circ$ - that satisfies these four axioms:

1. Closure: It's closed under $\circ$

$$(a \circ b) \in G, \quad \forall a, b \in G$$

2. Associativity: For any three elements (a, b, c) in G, the following equation holds:

$$(a \circ b) \circ c = a \circ (b \circ c)$$

3. There exists an element, e, in G such that the following equation holds for all elements in G:

$$e \circ a = a \circ e = a$$

4. For every element a in G, there exists an element a^{-1} in G (called the inverse of a) such that:

$$a \circ a^{-1} = a^{-1} \circ a = e$$

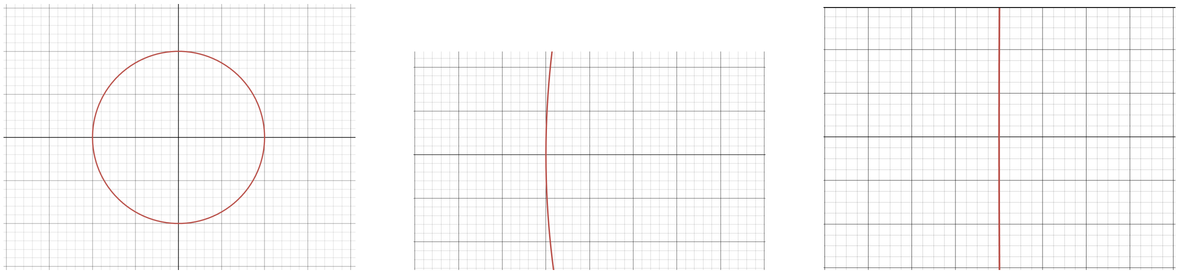
A good example of a group would be the set of integers ($\mathbb{Z}$) under addition (you can check satisfies our group axioms for yourself). Groups often represent physical symmetries though, and we are about to look at this aspect in more detail.

3 Circles are weird

Circles have rotational and reflectional symmetry. However, these symmetries are slightly different from those of shapes we are more used to. In a regular n-gon, symmetry operations are discrete, meaning there is a finite number of valid rotations and reflections that leave the shape invariant. Circles, contrastingly, look the same no matter how you orient them – they have infinitely many lines of symmetry and are invariant under rotation by any angle θ. These symmetries are continuous; there are no steep jumps between valid rotation angles. Because of this, it's difficult to study them in the same way group theorists study discrete ones: How could we accurately describe continuous symmetry, where there are infinitely many symmetry transformations?

4 Why are these groups lying to me??

Lie groups! Lie groups (pronounced 'Lee') are special types of groups that describe continuous symmetries. They are special because they are both groups and smooth manifolds. Manifolds are shapes that locally resemble $\mathbb{R}^n$. A simple example of a manifold is a circle; zooming in closely at any point on one, we see a straight line (1-dimensional euclidean space – $\mathbb{R}^1$)



(a) Full view of a circle (b) Zooming into leftmost point (c) Zooming in even closer

Figure 1: As we zoom into the circle, it looks more and more like a straight line - $\mathbb{R}^1$.

Hence, a circle is a 1-dimensional manifold. Another interesting fact is that the Lie group describing the rotational symmetries of a circle has the same topological properties as one - we'll look at this in more detail later.

5 Some properties of rotation matrices

Circles have infinitely many valid symmetry transformations (rotations and reflections). Reflections slightly complicate things, so I won't talk much about them in this essay. To better understand the transformations (and by extension Lie groups) I *will* talk about, it's useful to use linear algebra, which helps us see them in a more mathematical light.

Any 2D rotation, $\mathbf{R}$, θ radians counterclockwise about the origin can be represented by the standard (2x2) matrix:

$$\mathbf{R} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

An important general property of rotation matrices is that:

$$\mathbf{R}^T\mathbf{R} = \mathbf{R}\mathbf{R}^T = \mathbf{I}$$

We can show this property holds for any such matrix, R, like so:

$$\mathbf{R} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \quad \mathbf{R}^T = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

Compute $\mathbf{R}\mathbf{R}^T$:

$$\begin{aligned} \mathbf{R}\mathbf{R}^T &= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \\ &= \begin{pmatrix} \cos^2\theta + \sin^2\theta & \cos\theta\sin\theta - \sin\theta\cos\theta \\ \sin\theta\cos\theta - \cos\theta\sin\theta & \sin^2\theta + \cos^2\theta \end{pmatrix} \end{aligned}$$

Similarly, compute $\mathbf{R}^T\mathbf{R}$:

$$\begin{aligned} \mathbf{R}^T\mathbf{R} &= \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \\ &= \begin{pmatrix} \cos^2\theta + \sin^2\theta & -\cos\theta\sin\theta + \sin\theta\cos\theta \\ -\sin\theta\cos\theta + \cos\theta\sin\theta & \sin^2\theta + \cos^2\theta \end{pmatrix} \end{aligned}$$

We can rearrange to show that the two matrices are equal. Then, using the identity

$$\sin^2\theta + \cos^2\theta = 1,$$

we obtain

$$\mathbf{R}\mathbf{R}^T = \mathbf{R}^T\mathbf{R} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{I}.$$

Why does this matter? Well, all matrices that satisfy the above equation matter so much that they have a special name; they are orthogonal matrices! Orthogonal matrices are important in our exploration of symmetry because they possess unique properties. To investigate these properties, let's consider what happens when we apply orthogonal transformations to regular vectors.

6 Orthogonal matrices: What are they?

If we have some 2-dimensional vector $\mathbf{v}$ and apply the orthogonal transformation Q to it, we left multiply by the matrix representation of Q, obtaining the transformed vector $\mathbf{Q}\mathbf{v}$. Because $\mathbf{Q}$ is an orthogonal matrix, we know that $\mathbf{Q}\mathbf{Q}^T = \mathbf{Q}^T\mathbf{Q} = \mathbf{I}$. From this, we can deduce two properties that are the same before and after $\mathbf{v}$ is transformed.

Let $\mathbf{v} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$

Compute the length of $\mathbf{v}$, and the squared length of $\mathbf{v}$:

$$\|\mathbf{v}\| = \sqrt{x_1^2 + y_1^2}$$

$$\|\mathbf{v}\|^2 = x_1^2 + y_1^2$$

Note:

$$\begin{aligned} x_1^2 + y_1^2 &= \begin{pmatrix} x_1 & y_1 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \\ &= \mathbf{v}^T \mathbf{v} \end{aligned}$$

Similarly, let $\mathbf{Qv} = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$

Compute length of $\mathbf{Qv}$, and squared length of $\mathbf{Qv}$:

$$\begin{aligned} \|\mathbf{Qv}\| &= \sqrt{x_2^2 + y_2^2} \\ \|\mathbf{Qv}\|^2 &= x_2^2 + y_2^2 \end{aligned}$$

Again, note:

$$\begin{aligned} x_2^2 + y_2^2 &= \begin{pmatrix} x_2 & y_2 \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \\ &= (\mathbf{Qv})^T \mathbf{Qv} \end{aligned}$$

Simplifying further:

$$\begin{aligned} &(\mathbf{Qv})^T \mathbf{Qv} \\ &\qquad = \mathbf{v}^T \mathbf{Q}^T \mathbf{Qv} \\ &\qquad = \mathbf{v}^T \mathbf{Iv} \\ &\qquad = \mathbf{v}^T \mathbf{v} \end{aligned}$$

Hence:

$$\begin{aligned} \|\mathbf{Qv}\|^2 &= \mathbf{v}^T \mathbf{v} \\ &= \|\mathbf{v}\|^2 \\ \implies \quad \|\mathbf{Qv}\| &= \|\mathbf{v}\| \end{aligned}$$

The length of **v** is invariant under transformation **Q**, so orthogonal transformations preserve lengths. Similarly, (using the dot product) we could show that angles preserved - orthogonal transformations preserve both lengths and angles! So, when I say rotations are orthogonal (in a non-mathematical sense) I just mean they don't distort space - they only move points around.

7 Our first Lie group

Something useful we gain from our study of orthogonal matrices is a mathematical definition of what 'rotation' is. It's somewhat obvious that rotation matrices have a determinant of 1. Intuitively, this is because they preserve both lengths and the orientation of points. Our new fact is more important than it seems though, as it differentiates rotations from reflections - transformations that are also orthogonal but are not orientation preserving (they have a determinant of -1). Using these properties, we can define a rotation as an orthogonal transformation with determinant 1.

Examining the set of 2x2 rotation matrices, we can notice important things about them:

Firstly, if we perform one rotation, and then another, our result is still a rotation - it is still an element of our original set. This is intuitive, but we can prove it using our definition:

Let $\mathbf{R}_1$, $\mathbf{R}_2$ be rotation matrices.

Applying $\mathbf{R}_1$ then $\mathbf{R}_2$ is equivalent to left multiplication by $\mathbf{R}_2\mathbf{R}_1$

To prove $\mathbf{R}_2\mathbf{R}_1$ is a rotation matrix, we must show it's orthogonal ($(\mathbf{R}_2\mathbf{R}_1)^T(\mathbf{R}_2\mathbf{R}_1) = (\mathbf{R}_2\mathbf{R}_1)(\mathbf{R}_2\mathbf{R}_1)^T = \mathbf{I}$) and it's orientation preserving ($\det(\mathbf{R}_2\mathbf{R}_1) = 1$)

$\mathbf{R_1}$, $\mathbf{R_2}$ are rotation matrices

$\implies \mathbf{R_1}^T\mathbf{R_1} = \mathbf{R_1}\mathbf{R_1}^T = \mathbf{I}, \quad \mathbf{R_2}^T\mathbf{R_2} = \mathbf{R_2}\mathbf{R_2}^T = \mathbf{I}$

Compute $(\mathbf{R}_2\mathbf{R}_1)^T(\mathbf{R}_2\mathbf{R}_1)$:

$$\begin{aligned}(\mathbf{R}_2\mathbf{R}_1)^T(\mathbf{R}_2\mathbf{R}_1) &= \mathbf{R}_1^T\mathbf{R}_2^T\mathbf{R}_2\mathbf{R}_1\\ &= \mathbf{R}_1^T\mathbf{I}\mathbf{R}_1\\ &= \mathbf{R}_1^T\mathbf{R}_1\\ &= \mathbf{I}\end{aligned}$$

Similarly, compute $(\mathbf{R}_2\mathbf{R}_1)(\mathbf{R}_2\mathbf{R}_1)^T$:

$$\begin{aligned}(\mathbf{R}_2\mathbf{R}_1)(\mathbf{R}_2\mathbf{R}_1)^T &= \mathbf{R}_2\mathbf{R}_1\mathbf{R}_1^T\mathbf{R}_2^T\\ &= \mathbf{R}_2\mathbf{I}\mathbf{R}_2^T\\ &= \mathbf{R}_2\mathbf{R}_2^T\\ &= \mathbf{I}\end{aligned}$$

Hence, $(\mathbf{R}_2\mathbf{R}_1)^T(\mathbf{R}_2\mathbf{R}_1) = (\mathbf{R}_2\mathbf{R}_1)(\mathbf{R}_2\mathbf{R}_1)^T = \mathbf{I}$

Compute $\det(\mathbf{R}_2\mathbf{R}_1)$:

$$\begin{aligned}\det(\mathbf{R}_2\mathbf{R}_1) &= \det(\mathbf{R}_2) \times \det(\mathbf{R}_1)\\ &= 1 \times 1\\ &= 1\end{aligned}$$

Hence, $\det(\mathbf{R}_2\mathbf{R}_1) = 1$

$\implies \mathbf{R}_2\mathbf{R}_1$ is also a rotation matrix - our set is closed under matrix multiplication.

Secondly, matrix multiplication is associative.

Thirdly, when θ is 0, we get $\mathbf{I}$. If we compose this with any other rotation matrix $\mathbf{R}$, we obtain $\mathbf{R}$ ($\mathbf{IR} = \mathbf{RI} = \mathbf{R}$): it is the identity element.

Finally, we can prove that every rotation matrix has an inverse that is also a rotation matrix:

Let $\mathbf{R}$ be a rotation matrix

$$\begin{aligned}&\implies \mathbf{R}\mathbf{R}^T = \mathbf{R}^T\mathbf{R} = \mathbf{I}\\ &\iff \mathbf{R}^T = \mathbf{R}^{-1}\end{aligned}$$

$$\det(\mathbf{R}^{-1}) = \det(\mathbf{R}^T)$$

$$
\begin{aligned}
&= \frac{1}{\det(\mathbf{R})} \\
&= \frac{1}{1} \\
&= 1
\end{aligned}
$$

Hence, $\det(\mathbf{R}^{-1}) = 1$

Check $\mathbf{R}^{-1}$ is orthogonal:

$$
\begin{aligned}
\mathbf{R}^T &= \mathbf{R}^{-1} \\
\implies \mathbf{R}^{-1}(\mathbf{R}^{-1})^T &= \mathbf{R}^T(\mathbf{R}^T)^T \\
&= \mathbf{R}^T\mathbf{R} \\
&= \mathbf{I}
\end{aligned}
$$

Hence $\mathbf{R}^T$ is orthogonal

So $\mathbf{R}^T$ is a rotation matrix

$\mathbf{R}^T = \mathbf{R}^{-1}$

$\implies$ $\mathbf{R}^{-1}$ is a rotation matrix - every rotation matrix has an inverse that is also a rotation matrix.

The mathematics above shows that the set of (2x2) rotation matrices satisfies the 4 group axioms. So, it forms a group under matrix multiplication!

This group is called SO(2) - the special orthogonal group. ‘Orthogonal’ is used here because rotations are orthogonal transformations. ‘Special’ refers to the fact we are only considering orientation preserving orthogonal transformations (rotations not reflections). Including reflections leads to a larger symmetry group - O(2) - with a more complex structure.

Since it is a lie group, SO(2) is also a manifold. If we look at the unit circle in the complex plane, each point on it can be uniquely described by expression $e^{i\theta}$.

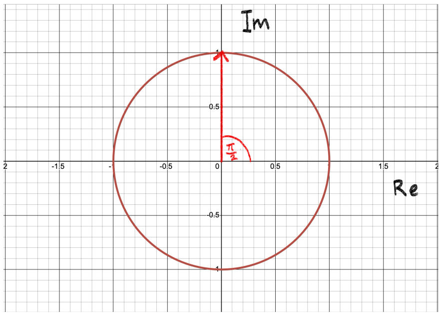


Figure 2: $e^{i\theta} = i\sin\theta + \cos\theta$ (Euler’s formula). This parametric graph plots $e^{i\theta}$ in the complex plane as we vary θ from 0 to 2π. The imaginary part is the y position and the real part is the x position. For example, when θ is $\frac{\pi}{2}$ we obtain $i + 0$ (or just i) which is visible in this diagram.

Similarly, in SO(2), each matrix can be uniquely described by the angle, θ, it rotates things by. We can pair every matrix in SO(2) with a unique point on the unit circle (specifically pairing so that θ of the rotation matrix is the same as θ in $e^{i\theta}$). This is called a mapping. An important property that *this* mapping has is that when we multiply two matrices together and multiply their paired coordinates on the circle (which are just two complex numbers) the resulting matrix and complex number also end up being pairs. This special correspondence between SO(2) and the unit circle is called an isomorphism. The unit circle is topologically a circle (of radius 1 but this doesn't matter) that has been embedded in $\mathbb{R}^2$. The fact that there is an isomorphism between SO(2) and the unit circle tells us that their topology is the same – SO(2) is a circle! This might be slightly confusing at first, but I'm not saying that SO(2) (a set of matrices) is *geometrically* a circle. It is *topologically* a circle – the set of points that form any circle and the set of matrices in SO(2) share specific properties. In an intuitive sense, SO(2) can be parameterised by θ (if we know θ we can work out the corresponding rotation matrix without uncertainty). If we imagine θ as a point on some line, it can take any value, *but* $2\pi + \theta$ corresponds to the same rotation matrix as θ; the line must 'loop' back on itself somehow, like a circle. This is super cool - we've just learned that SO(2) (the group that encodes rotational symmetries of a circle) kind of is one.

All the matrices in SO(2) are 2x2 – they describe 2-dimensional transformations. What happens when we look at rotations in higher dimensions?

8 Going into the 3$^{\text{rd}}$ dimension

3D rotation matrices also form a group under matrix multiplication – SO(3) (in fact, any set of all nxn rotation matrices will form a group under matrix multiplication)! They correspond to the rotational symmetries of a sphere. The algebraic properties of 3x3 rotations are the same as 2x2 ones as they are also orthogonal. Despite this, the topology of SO(3) is much more interesting than that of SO(2).

Firstly, a rotation in 3D is more complex than one in 2D. To describe such rotations, we first need to pinpoint the rotation axis.

Imagine, for example, you are holding a round ball and facing the screen of a computer. If you rotate the ball clockwise, the *axis* of this rotation is the vector pointing directly into the computer screen (from the ball's center). The points on this vector are invariant and the other points on the ball rotate around it.

Figure 3: If we rotate the ball in the direction of the blue arrow, the points that coincide with the red arrow won't move (so it is the rotation axis) - the other points on the ball will move *around* this arrow.

Next, to fully describe the transformation, we must specify how much we rotate around this axis. If you curl the fingers in your right hand and point your thumb in the direction of the rotation axis, the way your fingers are curled indicates the direction of a positive rotation about that axis. So, if we flip the direction of our axis (make your thumb point in the opposite direction), the 'positive rotation' direction is subsequently inverted.

Figure 4: The rotation axes (red arrows) are pointing in opposite directions, so +ve rotation direction (blue arrows) are flipped.

For all rotation axes, we only need to consider angles $0 \leq \theta \leq \pi$ because rotating through any θ ($\pi < \theta \leq 2\pi$) is equivalent to rotating through $2\pi - \theta$ about the flipped axis of rotation.

To visualise the topology of this group, we can imagine θ as a parameter telling us how far in some direction we've traveled from the origin. The interesting bit is there are infinitely many directions to travel in, corresponding to the infinitely many axes of rotation in 3D. Defining 3D rotations the way we have is useful because it means each rotation can be mapped to a unique point in this 3D space – we can say any given rotation is on some rotation axis (the direction in which we travel from the origin) and is θ along it. The resulting space is a ball (radius π) with infinitely many points inside it (similar to a solid sphere). Unlike other angles though, rotating π about some rotation axis is equivalent to rotating π about the corresponding inverted rotation axis (same vector pointing in the opposite direction). This means that the points in our space that these rotations are mapped to – points on the ball's surface that are polar opposites (antipodal points) – would have to be identified as the same point, since they correspond to the exact same rotation. This fact changes the topology of the space – it is no longer some solid sphere but a new space: $\mathbb{RP}^3$ (real projective 3-space).

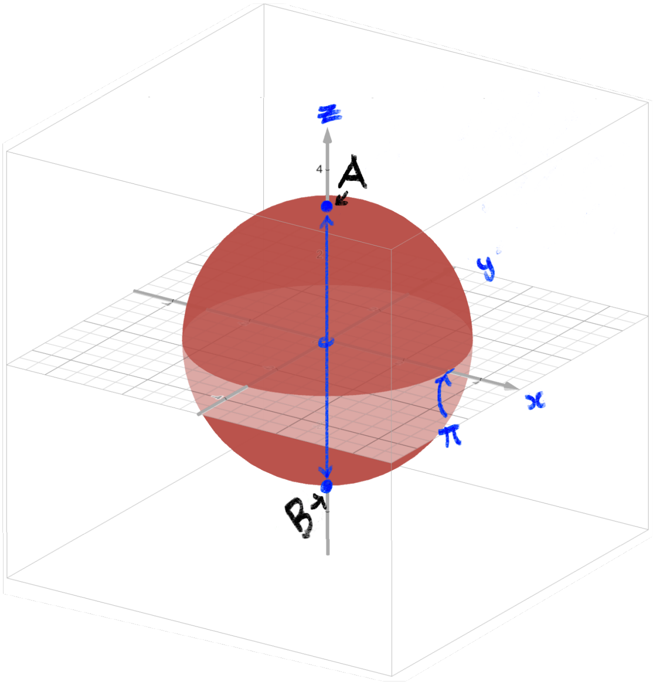
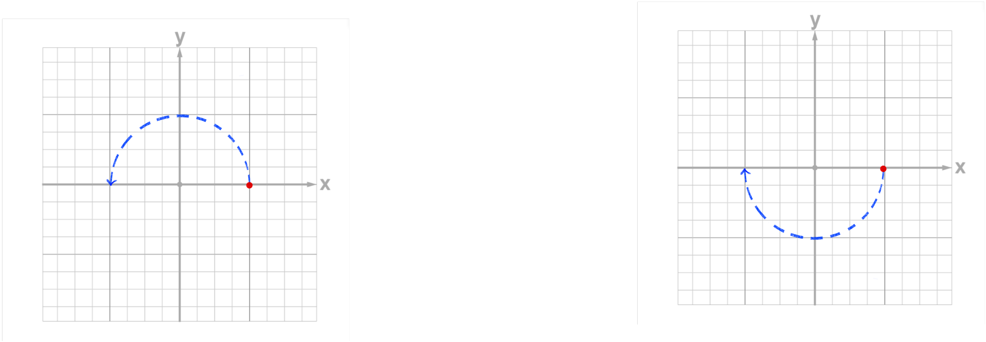


Figure 5: Every SO(3) rotation maps to a point in this space so can be described by a position vector. The magnitude of each position vector is the angle of rotation. The direction of each position vector is the axis of rotation. Points A and B both have magnitude π, but A is in +z direction and B is in the -z direction.



(a) A: corresponds to rotating through π about z axis (pointing out of the screen)

(b) B: corresponds to rotating through π about z axis with opposite orientation (pointing into the screen)

Figure 6: We can see A and B are the same transformation - this true for all antipodal points on the sphere meaning they must be identified as the same point.

9 Conclusion.. or just the beginning?

What starts as a simple idea - viewing symmetry as invariance - leads to groups like those we have looked at in this essay. Something I find beautiful about group theory is that despite how abstract these groups may seem, they are often grounded in reality. For example, the Lie groups I've talked about are connected to the universe's fundamental forces. In closing, I hope my essay has helped convey how rich group theory is. These groups are just the tip of a tall iceberg concerning questions at the heart of our universe - a universe built on symmetry.

10 References

Graphs made in Desmos, diagrams all my own.

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