

Dawn to Dusk: Modelling the Ordinary with Calculus

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Introduction

Welcome dear reader, excited to have you here!

Whenever you hear the term “*calculus*”, certainly many of us, in fact ***all*** of us have been **haunted** by this term once. The awkward derivatives, mind boggling integrals and the threats of the iconic limits. Mathematics and especially Calculus can be seen as incredibly abstract and difficult to understand its applications in the way we are taught in schools. However, with this essay, I really aim to show you what a wonderful branch of Mathematics this really is. My personal aim is to show that calculus is beautiful not through the conventionally complex theorems and problems, which require a great level of mathematical maturity (however elegant and **beautiful** they are) but through the way calculus is applied to the real world. Not only in fields such as Theoretical Physics, Economics and Data Science, but by taking an even larger step and showing these applications are present within simple activities in our daily lives. I want to demonstrate how all of us use calculus indirectly within our day, through this essay I hope to build and share with you reader, my deep admiration for this area of Pure Mathematics.

This essay will be structured in chronological order of the typical day of an average high school student (much like I) and how these seemingly simple activities that I and almost everyone conduct in our daily life , can be made much more complex and modelled with Mathematical concepts to predict change and optimisation.

The contents of this essay may seem unconventional but trust me, each section will reveal the hidden role of Mathematics within, particularly calculus.

(exciting!)

1. What is *Calculus*? And its history
2. Modelling early morning rates of energy
3. How long until my coffee gets cold?
4. Volumes of chocolates
5. Video game success

1) What is Calculus? And its history

Before we dive into the adventure into our typical day, we must summarise and go through the basics. What Calculus really *means*. Thus diving into its deeply interesting history.

In simple words Calculus is the math of change and totals. It helps you understand how things change (speed, population etc). Introducing the fundamental branch of differential calculus.

The other interpretation is through the branch of integrals, existing as the antiderivative (opposite of differentiation). It not only allows us to discover functions based on their derivatives, but also has an application of discovering totals (such as area and volume).

The ideas of Calculus date back to ancient civilisations. Greek Mathematicians like Archimedes used early ideas of limits to find areas and volumes, whilst scholars in India and the Islamic world made important advances in infinite series and algebra.

However, the two fundamental pillars and the greatest contributors to this magnificent field are Sir Isaac Newton and Gottfried Wilhelm Leibniz



These elegant and excellent mathematicians allowed the development of calculus in their own ways

Isaac Newton revolutionised mathematics in the 1660s by developing the 'method of fluxions' (differential calculus) to solve problems in theoretical physics involving problems with motion, rates of change and curvature. Developing the fundamentals of integration and differentiation applying them to planetary orbits, proving the connection between calculus and physics

On the other hand Leibniz also contributed to calculus in a different way. Whilst both Newton and Leibniz developed differential and integral calculus. Leibniz was

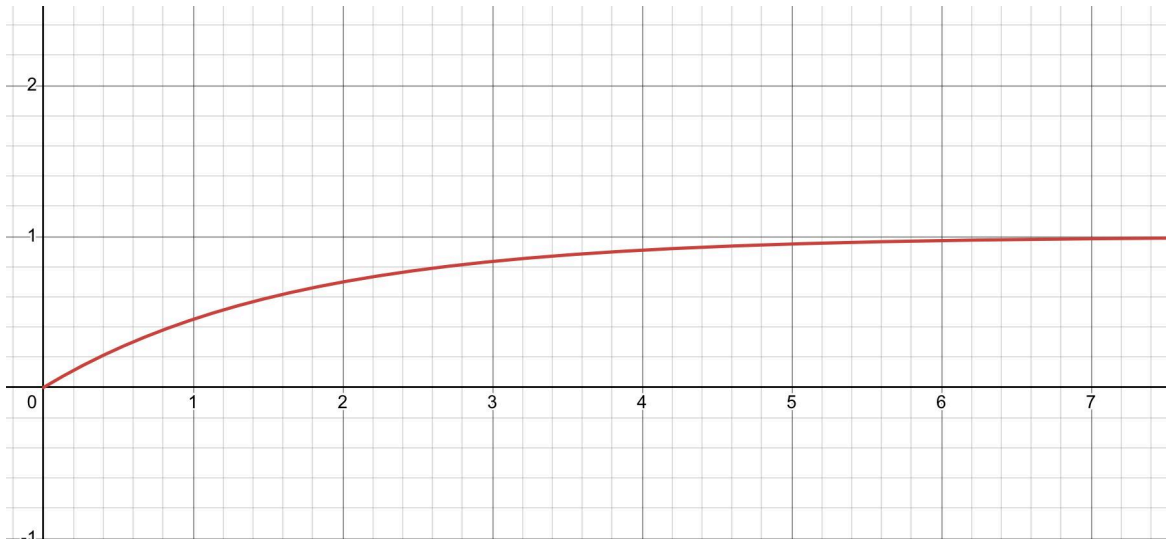
responsible for introducing the key notations such as the integral (\int) and the differential of dy/dx . He additionally discovered many other differential rules such as the product rule and the fundamental theorem of calculus that crucially argued that integration and differentiation are inverses.

With that we conclude our history lesson and begin our incredibly mathematical and complex day of our high school Student. For avoiding repetition and more simplicity we shall name him "Bob".

2) Modelling early morning rates of energy.

So we begin our day! Lazy Bob is still in bed and late for school. He rises swiftly as the alarm repeatedly goes off near him. As Bob gets ready for school, he effectively wonders the true nature of our energy levels as soon as we wake up. It is a social convention and widely understood that as we wake up, our energy starts off at a minimum, increasing over a period of time and levelling off eventually. Bob, like us, a curious mind entrapped with the magic of Calculus, wonders how he could mathematically model this particular situation using Calculus.

First and foremost, he decides we need a function which roughly demonstrates the behaviour of our energy levels with respect to time. As our energy increases more rapidly as time goes on he settles to using the Euler constant of 'e' modelling the exponential growth of energy. In order to model the "levelling off" he reaches to his function $E(t)$ to be defined as $E(t) = 1 - e^{-kt}$ defined for t (time) is greater than or equal to 0 (our observing of energy levels start at waking up thus cannot have a "negative" time) and where 'k' is a real constant. Let us use $k=0.6$ to model this scenario



$$1 - e^{-0.6t} \{t \geq 0\}$$

Bob uses Desmos here to generate a graph of his energy levels, and settles upon the following function above.

Now Bob wonders, at what time is his energy growing at the fastest rate?

Bob uses his new profound knowledge of calculus to figure this out.

Hers a picture of his notebook:

To find where energy is growing the fastest we aim to find the derivative

$$E(t) = 1 - e^{-0.6t}$$

$$E'(t) = 0.6 e^{-0.6t}$$

$$t=0 : E'(t) = 0.6$$

$$t=2 : E'(t) = 0.1807...$$

$$t=3 : E'(t) = 0.099...$$

to find my "levelled off" energy increase

$$= \lim_{t \rightarrow \infty} 0.6 e^{-0.6t} = 0.6(0) = \underline{\underline{0}}$$

\therefore Thus my energy increases the fastest at time $t=0$, and stops increasing as $t \rightarrow \infty$

As you can see, due to Bob finding his derivative of the energy function, and plugging in values for 't' he is able to suggest and prove Mathematically using calculus, that his energy increase is the greatest after waking up. Additionally by taking the limit, he again proves using calculus that his energy increase as time grows larger and larger begins to decrease to 0, allowing him to reach his maximum energy.

Now, Bob wishes to find a quantitative measure of how much energy he accumulates given two periods of time. Now, as it is very difficult to measure energy directly Bob results to quantify this quantity as the area under the curve.

Here's another page:

$$\begin{aligned}
 & \text{My accumulation of energy in the first 5 minutes?} \\
 & \text{energy accumulation} = \int_0^t E(t) dt \\
 & \text{in } t \text{ minutes} \\
 \Rightarrow & \int_0^5 1 - e^{-0.6t} dt \\
 & \left[t + \frac{5}{3} e^{-0.6t} \right]_0^5 \\
 & \left(5 + \frac{5}{3} e^{-0.6(5)} \right) - \left(0 + \frac{5}{3} e^{-0.6(0)} \right) \\
 & \quad 5.083 - 1.6667 \\
 & = \underline{\underline{3.4163}} \\
 & \text{my energy accumulated in the first five minutes is } \underline{\underline{3.4163}}
 \end{aligned}$$

Here Bob is able to successfully get an accurate measure for this accumulation by integrating over his desired time periods. Thus being able to compare how his energy accumulation varies throughout his mornings. However, like any modelling scenarios, Bob has made many assumptions.

This model must only be used in the times of early mornings, this is because as time grows larger and larger, people tend to become tired and are unable to stay at maximum energy output. Additionally any food or drink can largely impact his rate of change, allowing us to question if Bob's model really is useful for providing a measure of energy levels, or is it an excuse for him to do calculus in his mornings (relatable).

3) How long until my coffee gets cold??

Now that Bob is awake and at his "leveled" off maximum energy level, he aims to get a quick bite before heading off to school. To increase his energy level further he pours himself a cup of coffee (breaking his previous assumption). However, his coffee is scorching hot. Thus impatient Bob gets a thermometer, pen and a notebook to figure out when he's able to get his daily caffeine.

using my thermometer, I find } conditions
 Initial temp = $T_0 = 90^\circ\text{C}$
 Room temp = $T_{\text{room}} = 20^\circ\text{C}$
 after 5 minutes = 60°C }
 "Newton's cooling law"
 $\frac{dT}{dt} = -k(T - T_{\text{room}})$ ← Differential equation
 $\frac{dT}{dt} = -k(T - 20)$
 $\int \frac{1}{T-20} dT = \int -k dt$ ← integrating both sides
 $\ln|T-20| = -kt + C$ ← raise to the power of "e"
 $T - 20 = Ae^{-kt}$ ← General solution to the differential equation
 $T(t) = 20 + Ae^{-kt}$
 initial
 $90 = 20 + Ae^{-k(0)}$ ← use initial condition to find constant "A"
 $90 = 20 + A$
 $70 = A$
 $T(t) = 20 + 70e^{-kt}$ ← use $T(5)$ to find constant k
 $60 = 20 + 70e^{-5k}$
 $40 = 70e^{-5k}$
 $\frac{4}{7} = e^{-5k}$
 $\ln\left(\frac{4}{7}\right) = -5k$

$-\frac{1}{5} \ln\left(\frac{4}{7}\right) = k$ ← function of temperature in relation to time
 $0.112 \approx k$
 $T(t) = 20 + 70e^{-0.112t}$
 I like my coffee at 50°C , how long should I wait?
 $50 = 20 + 70e^{-0.112t}$ ← Set function = 50
 $30 = 70e^{-0.112t}$
 $\frac{3}{7} = e^{-0.112t}$
 $\ln\left(\frac{3}{7}\right) = -0.112t$ ← solve!
 $\frac{\ln\left(\frac{3}{7}\right)}{-0.112} = t$
 $7.565 \approx t$
 \therefore I should wait 7.565 mins before I drink my coffee !!

Here's two pages of his findings,

Bob here, recognises and remembers the "Newton's Cooling Law". The Law that the rate of change of temperature with respect to time is proportional to the temperature of the coffee and the surrounding temperature. Allowing him to solve a differential equation, leading to a temperature function in terms of time. Thus he is able to find the time taken for his coffee to reach his desired temperature easily. "Neat!" he yells to himself as he enjoys his caffeine.

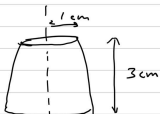

The assumptions here he has made are much more feasible than the last model, thus making it far more useful.

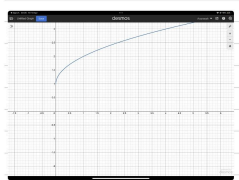
These include:

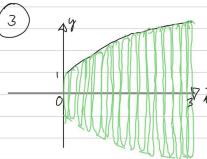
1. Constant room temperature
2. Constant cooling constant
3. Every part of the coffee is the same temperature


4) Volumes of chocolates

Now that Bob is well rested and energised with his coffee, he quickly hurries to his secondary school. After a day of devastating and boring lessons (except Math) he rejoices as he finds out the next period is lunch! Bob quickly scurries toward the lunch hall and sees the sign "Chocolate Fridays". Bob hurries and grabs a handful, upon opening them, the chocolates seemed unsatisfactory to him, so he took on a mission to figure out their volumes and show his classmates their insufficient nature. So he grabs a ruler, measures the lengths of the chocolates and then represents it on a graph in the form of function.

①  

②  → using measurements to plot the graph
 $y = \sqrt{x} + 1$

③  360 degree rotation about x axis

④  each "disc" could be seen as a cylinder with radius y and height δx (infinitesimally small change in x)

⑤ Volume of one "disk" = $\pi y^2 \delta x$

We aim to find the volume generated of all the disks, from the x values 0 to 3

$$\therefore V = \sum_{x=0}^3 \pi y^2 \delta x$$

⑥ SOLVE!

$$V = \int_0^3 \pi y^2 dx$$

$$y = \sqrt{x} + 1$$

$$y^2 = (\sqrt{x} + 1)^2$$

$$y^2 = x + 2\sqrt{x} + 1$$

$$V = \pi \int_0^3 (x + 2\sqrt{x} + 1) dx$$

$$V = \pi \left[\frac{1}{2}x^2 + \frac{4}{3}x^{3/2} + x \right]_0^3$$

$$V = \pi \left(\left(\frac{1}{2}(3)^2 + \frac{4}{3}(3)^{3/2} + (3) \right) - \left(\frac{1}{2}(0)^2 + \frac{4}{3}(0)^{3/2} + (0) \right) \right)$$

$$V = \pi (14.4282)$$

$$V = 45.3275$$

$$V = 45.3 \text{ cm}^3$$

Neat! Bob used the classic trick of "Volumes of revolutions" mapping the general shape of the chocolate in the form a function and rotating it 360 degrees about the x axis, conveniently letting him work out the volume generated by the individual chocolates

The assumptions used in this model include:

1. Chocolates required to be smooth
2. Chocolate required to have a vertical line of symmetry
3. The shapes of the chocolates may vary, leading the volume to be unrepresentative of the other chocolates.

5) Video Game success

Bob now, after successfully convincing his friends of the lacking chocolates, arrives home and decides the final task of the day for him is going to be playing Minecraft. A sandbox game, with a blocky world, allowing players to explore and build anything the imagination craves. He looks upon his base and sees an area which could be useful as a chicken farm. However he has been given a limited amount of fence blocks to use and wants to maximise the area produced. Bob gets the brilliant idea of finding an equation which relates to the area and optimises it by using differentiation to find the dimensions of the farm (length and breadth)

* I have 75 fences

① Area = $L \times W$

② $75 = 2L + 2W$
 $75 - 2L = 2W$
 $\frac{75}{2} - L = W$

Plug into equation one
Area equation in terms of length

$$A = L \times \left(\frac{75}{2} - L \right)$$
$$A = \frac{75}{2}L - L^2$$
$$\frac{dA}{dL} = \frac{75}{2} - 2L$$

Graph of the Area equation

Area is maximised at its maximum turning point, thus $\frac{dA}{dL} = 0$

③ $0 = \frac{75}{2} - 2L$

$$-75 = -2L$$
$$\frac{75}{2} = L$$
$$18.75 = L$$

↓ maximum length

$75 = 2(18.75) + 2W$
 $75 = 37.5 + 2W$
 $37.5 = 2W$
 $18.75 = W$

plug "L" into equation
→ maximum width

as the fence blocks must be whole number we round down

Actual maximum length and width = 18

$$\text{max Area} = 18 \times 18$$
$$= \underline{\underline{324 \text{ blocks}^2}}$$

Conclusion

Congratulations reader! Successfully reaching the end of our typical calculus filled day of Bob. From waking up in the morning, to drinking his daily caffeine content, to analysing chocolates with precision and finally ending the day off with a hearty game of Minecraft.

I hope through our character and his Mathematical day you can now have a sense of appreciation for not just calculus, but every area of Mathematics. I wanted to demonstrate the fundamental usefulness of Mathematics in a diverse range of areas. Not just the common cases of complex sciences such as Physics but also its effective usage in something as simple as finding the volume of a chocolate!

Now, it is obvious, that certainly many of us aren't like Bob and we don't usually use Calculus for areas such as monitoring energy accumulation, but I hope to demonstrate that through these highly simple examples , we can truly appreciate the art of Mathematics in itself , how it exists almost in every scenario, existing in nature, waiting to be discovered.

I wish this essay kept you engaged! Hoping that with your now new appreciation of this branch of Mathematics, you are ambitious to explore deeper. Maybe the magic of Multivariable Calculus, or perhaps aiming to learn new techniques, allowing for faster and more efficient processes, such as the Feynman Technique (exciting!), or Contour integration.

May the force of Mathematics be with you!