The Geometry of Social Distance: Proving the Kissing Number of Two-Dimensional Space
(Written by Devon Richard)

Introduction

Imagine standing in an open field or a park, and imagine drawing an invisible circle around yourself with a radius of exactly two metres. This is the familiar "social distance" boundary we became so accustomed to not long ago—a protective bubble defining your personal space. Now, ask yourself a simple question: how many other people can stand around you such that each of them is exactly two metres away from you, and also exactly two metres away from their neighbours? How many such circles can fit touching yours without overlapping?

At first glance, this might seem like a trivial problem of common sense. But look closer, and you realise you have stumbled upon one of the oldest and most elegant problems in geometry: the Kissing Number Problem. It asks: what is the maximum number of non-overlapping spheres that can touch another sphere of the same size?

In two dimensions, where spheres become circles, the answer is surprisingly elegant. You can fit exactly six. No more, no less. It is impossible to squeeze a seventh.

In this essay, we will explore this fascinating result. We will look at its history, dating back to a famous debate between Isaac Newton and David Gregory. We will construct a rigorous proof using nothing more than basic high-school trigonometry and logic. Finally, we will see how this simple fact about circles connects to the structure of honeycombs, the design of cities, and even the way we send data across the internet.

What is the Kissing Number?

In mathematical terms, the kissing number in two dimensions, denoted as $k(2)$, is defined as the maximum number of non-overlapping unit circles that can be placed in the same plane so that each one touches a common central unit circle. The term "kissing" refers simply to the point of contact, where the circles touch but do not cross.

You can visualise this easily using coins. If you place one coin flat on a table, how many other identical coins can you push against it so that they all touch the centre one, but none of them overlap each other? You will quickly find you can arrange six around it, but the seventh simply will not fit.

This problem is not new. Nature has known the answer for millions of years. Bees construct their honeycombs in hexagonal patterns precisely because this arrangement uses the least amount of wax to hold the most honey—it is the most efficient way to pack circles together.

However, the formal history of the problem is quite interesting. It began in 1694 with a correspondence between Isaac Newton and the astronomer David Gregory. Newton correctly believed that only 12 spheres could touch a central sphere in three dimensions, while Gregory argued that perhaps 13 might fit. While they were arguing about 3D, the 2D case was the foundation of their discussion. Although the answer for 2D seemed obvious, a truly rigorous proof took time to appear. It was not until 1892 that Axel Thue provided the first proper proof, which was later refined significantly by László Fejes Tóth in the 1940s.

Why does this matter today? Beyond coins and honeycombs, these packing problems are vital in modern science. They appear in coding theory, where mathematicians pack "messages" into digital space, and in crystallography, where scientists study how atoms arrange themselves into lattices. Understanding how things fit together is fundamental to understanding the physical world.

Setting Up the Geometry

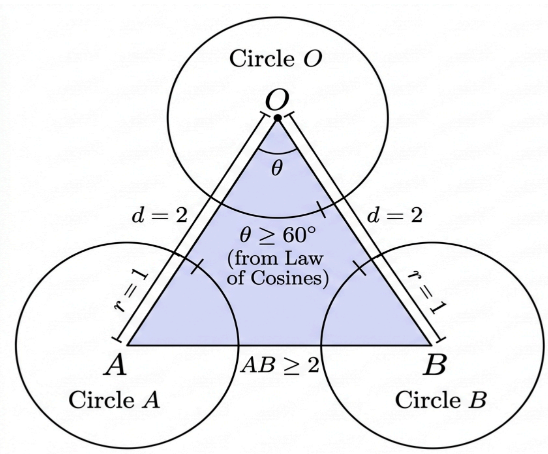

Figure 1: Illustration of the isosceles triangle formed by the center O and the centers of two surrounding circles A and B. Since all circles have radius 1, the distances OA and OB are exactly 2 units. For the circles not to overlap, the distance AB must be at least least 2 units, which by the of Cosines implies that the angle θ at the center must be at least 60°.

To prove that six is the maximum, we need to move away from coins and look at the geometry precisely.

Let us formalise the situation:

- We have a central circle, let's call it $C0$, with a radius of 1 unit.
- We want to place other circles $C1$, $C2$,........., Cn around it. These are also radius 1.
- The rules are strict: every outer circle must touch $C0$, and no two outer circles are allowed to overlap. They can touch, but they cannot cross.

Here is the key insight: the circles are defined by their centres. If we know where the middle of each circle is, we know everything.

Since every outer circle touches the central one, the distance between the centre of an outer circle and the origin must be exactly 1 + 1 = 2 units. This means that if we were to draw dots only at the centres of all the outer circles, they would all lie exactly on a larger circle of radius 2 surrounding the origin.

Now, what about the distance between two neighbouring outer circles? If they are not allowed to overlap, the distance between their centres must be at least 2 units. If it were any less than 2, the circles would overlap.

So, we have a triangle formed by:

1. The origin (centre of $C0$).
2. The centre of circle $C1$.
3. The centre of circle $C2$.

Two sides of this triangle are length 2. The third side must be at least length 2.

We can use the Law of Cosines to find the angle at the origin between these two circles. The law states:

$c^2 = a^2 + b^2 - 2abcos(\theta)$

We know $a = 2$, $b = 2$, and we require $c \geq 2$.

$(distance)^2 \geq 2^2$

$2^2 + 2^2 - 2 \times 2 \times 2cos(\theta) \geq 4$

$8 - 8cos(\theta) \geq -4$

Rearrange this gives us:

$-8cos(\theta) \geq -4$

$cos(\theta) \leq 0.5$

Now, we look at the cosine values. When is $cos(\theta)$ equal to 0.5? Precisely when $\theta = 60^0$. And since cosine gets smaller as the angle gets bigger, this means that the angle θ must be at least 60^0.

This is our golden rule. If you want to place two circles around the middle one without them crashing into each other, they must be separated by an angle of at least 60 degrees as viewed from the very centre.

The Proof by Contradiction

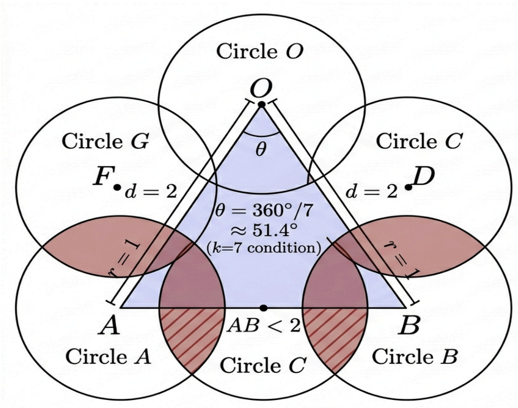

Figure 2: Attempting to place seven circles around a central circle. Evenly spaced, the angle between neighbors is only $360°/7 \approx 51.4°$. Since this is less than the required minimum of 60°, the distance between centers becomes less than 2, causing the circles to overlap (shown by the shaded intersection area), proving that $k(2) < 7$.

Now we are ready to prove that seven is impossible. We will use a method called Proof by Contradiction. This means we will assume the opposite of what we believe is true, and then show that this assumption leads to nonsense or an impossibility.

1. **Step 1: The Assumption**

Let us suppose, for the sake of argument, that it is possible to place seven circles around the central one.

2. **Step 2: The Angular Requirements**

From our previous calculation, we know that every single pair of neighbouring circles requires a minimum angle of 60^0 to fit without overlapping.

If we have seven circles, we need seven of these angles.

$7 \times 60^0 = 420^0$

3. **Step 3: The Contradiction**

However, we know from basic Euclidean geometry that a full rotation around a point is exactly 360^0. We have hit a contradiction. You cannot pack 420^0 of "angular requirements" into a 360^0 space. It simply does not fit.

Some might argue: "But what if we don't space them out evenly? What if we squeeze some closer together so we can make more room?"

This is where the Pigeonhole Principle comes into play. Imagine you have 360 degrees to distribute as "gaps" between 7 circles. The average gap size would be $360/7\ approx.\ 51.4^0$. Mathematically, it is impossible for all the gaps to be larger than the average. Therefore, at least one of those gaps must be 51.4^0 or smaller. But remember our rule: any gap smaller than 60^0 is too small. If two circles are only 51.4^0 apart, the distance between their centres would be less than 2 units, meaning they overlap. Therefore, our initial assumption—that seven circles can fit—is wrong. It is mathematically impossible.

4. **Step 4: Showing that Six Works**

Of course, proving that seven is impossible isn't quite enough; we must also show that six is actually possible. If we place six circles, the math works out perfectly.

$$6 \times 60^{0} = 360^{0}$$

It fits exactly. We can place the centres at the vertices of a regular hexagon. In this configuration, every angle is exactly 60^{0}, and the distance between neighbours is exactly 2 units. The circles touch perfectly—they kiss their neighbours and they kiss the centre—but nobody overlaps. This configuration is "tight". There is no wasted space. And thus, we have our answer: The kissing number $k(2) = 6$.

The Social Distance Analogy

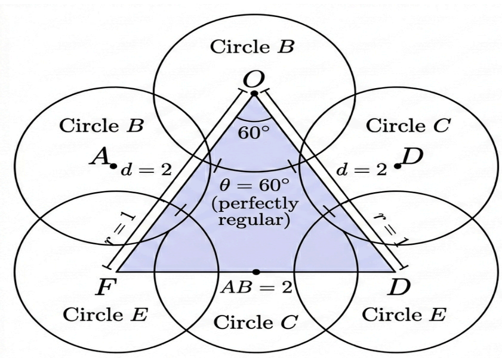

Figure 3: The successful configuration showing $k(2) = 6$. The centers form a perfect regular hexagon. Every angle is exactly 60°, allowing the six surrounding circles to touch the central one and their immediate neighbors perfectly without any overlap. This hexagonal packing is the densest possible arrangement of circles in a plane.

Let us translate this back to the real world and the image we started with. If we model people as points, and the 2-metre safety rule as a radius around them, our proof gives a hard limit on human interaction. If you are standing in that park, you can invite at most six friends to stand around you and maintain the exact 2-metre distance from everyone.

If a seventh friend arrives and wants to join the inner circle, the geometry breaks. One of three things has to happen:

1. Two people will end up closer than 2 metres to each other (violating the rule).
2. Someone will have to stand further away than 2 metres from you (breaking the "inner circle").
3. You leave the 2D plane entirely (like hovering in 3D space).

Now, humans are not perfect circles or points, and social distancing is a guideline, not a rigid physical law like atomic repulsion. However, this "circle packing" model is incredibly useful. It is used in urban planning, simulation of crowd dynamics, and even logistics. This geometry explains why hexagonal arrangements are so powerful. When you pack circles so that every circle has exactly six neighbours, you achieve the highest possible density. Mathematically, these circles cover about 90.6% of the available area. You cannot pack identical circles any tighter than that, a theorem first proven by Axel Thue. This is why nature loves hexagons—from the eyes of flies to the cells of beehives—it is the ultimate efficiency.

Why Pure Geometry Matters?

Proving that $k(2) = 6$ might seem like a small victory, perhaps even obvious. But it serves as the gateway to some of the most complex and important problems in modern mathematics. As we move into three dimensions, the problem gets much harder. In 3D, the kissing number is 12, but interestingly, there is actually leftover space. Unlike the 2D case where everything fits like a jigsaw puzzle, in 3D you cannot fit a 13th sphere, but you are left with gaps that are tantalisingly large. This was exactly the confusion that Newton and Gregory argued about centuries ago.

The problem becomes truly mind-bending when we go to higher dimensions. Why does this matter? Because this is where Coding Theory lives.

When you send a message over the internet or via satellite, it is converted into a string of numbers. Mathematicians represent these messages as "points" in a high-dimensional space. To prevent errors or static from corrupting the message, we place these points far apart from each other, effectively surrounding them with "spheres" of error tolerance.

The question then becomes: how many such messages can we send before their "error spheres" start overlapping? This is exactly the kissing number problem!

Remarkably, we only know the exact answer for a handful of dimensions. We know it for 1D, 2D, 3D, 4D, 8D, and 24D. The solutions for 8 and 24 dimensions are particularly beautiful, relying on special structures called the $E8$ lattice and the Leech lattice. In 24 dimensions, the kissing number is a staggering 196,560. For almost every other dimension, we are still guessing, working with upper and lower bounds.

Conclusion

From placing coins on a table to designing the algorithms that run the internet, the kissing number problem defines the fundamental limits of space. By examining the geometry of "social distancing", we discovered that in two dimensions, the number six is not just a suggestion, but a mathematical necessity. It is written into the laws of geometry. We saw how a simple question led us through history, from Newton to Thue, and used basic trigonometry to create an unbreakable proof. We saw that you simply cannot fit seven circles where six belong, because 51.4^0 is just not enough angle.

This journey reminds us that mathematics is far more than a list of rules to memorise. It is the underlying structure of reality. Whether you are trying to organise a safe gathering in a park, understanding why a honeycomb is shaped the way it is, or sending secure data across the globe, you are bound by these same geometric truths. Geometry turns our intuition into certainty, ensuring that no matter how hard we try, the 360^0 of the circle will always have the final say.

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