

Mind-Boggling Numbers

The Biggest of the Big

Dilan Patel – London, UK

Introduction

Let's start with a simple question. What's the biggest number you can think of?

Some might say one million, one billion or one trillion. Of course, in the grand scheme of things, these numbers are small. But how small really are they? How big can we go? Here, we will take a glimpse into some of the biggest numbers out there, and some of their applications, gradually growing from googol to Graham's Number and beyond.

Before we begin, it is important to realise that these numbers are overwhelming for the human mind to comprehend (Numberphile, Graham's Number - Numberphile, 2012). Comprehend these numbers at your own risk.

Operations and Notation to Build Large Numbers

Firstly, we will look at the processes and operations required to create large numbers. Explained below are the following operations, which increase in magnitude:

- Succession (increasing a number by 1)
- Addition (repeated succession)
- Multiplication (repeated addition)
- Exponentiation (repeated multiplication)

As can be seen, each new operation is a repetition of the last, making it a more powerful growth process (Genius, 2023).

To demonstrate this, we will take a positive integer. Here, we will use 3. We will perform the operation of succession, as shown below.

$$3 + 1 = 4$$

Now, we shall perform addition by adding 1 three times. This can also be written as

$$3 + 1 + 1 + 1 = 3 + 3 = 6$$

Next, we will perform repeated addition, which is known as multiplication. This is shown below.

$$3 + 3 + 3 = 3 \times 3 = 9$$

Going even further, we can repeat this multiplication to show the process of exponentiation, as shown.

$$3 \times 3 \times 3 = 3^3 = 27$$

As we will see later, this can also be written as $3 \uparrow 3$. This notation will be introduced in the Graham's Number section. The concept of using operations of increasing magnitude can be continued indefinitely – the next three operations in the sequence are

tetration, pentation and hexation. At this stage, these operations become known as ‘hyperoperations’.

As we can see, from the results of carrying out the operations, each new operation grows faster than the last. This is shown in Table 1.

| Operation | Result | | |
|----------------|--------|-----|------|
| | 3 | 4 | 5 |
| Succession | 4 | 5 | 6 |
| Addition | 6 | 8 | 10 |
| Multiplication | 9 | 16 | 25 |
| Exponentiation | 27 | 256 | 3125 |

Table 1 – Operations in increasing order of magnitude

Specifically, as the magnitude of the operation increases, the growth rate increases, not just the result. Therefore, hyperoperations are commonly used to build larger numbers.

Aside from this process, we can introduce factorials (written as $n!$), a form of repeated multiplication. The basic principle is shown below.

$$n! = n \times (n - 1) \times (n - 2) \times \dots \times 2 \times 1$$

In other words, n is multiplied by all of its preceding positive integers, down to 1. Shown in Table 2 are a list of the first few factorials.

| N | $n!$ |
|---|------|
| 1 | 1 |
| 2 | 2 |
| 3 | 6 |
| 4 | 24 |
| 5 | 120 |
| 6 | 720 |

Table 2 – The first six factorial numbers

It is clear that the factorial operation allows numbers to grow extremely quickly, and it can be used to construct huge numbers. This is because each new number is multiplied by a larger number than before. For example, $60!$ ($\approx 10^{82}$) is a good approximation for the number of atoms in the observable universe (RealLifeLore, 2020).

the pentagon as $pentagon(n) = n$ inside n squares and so on. This is shown in Figure 2.

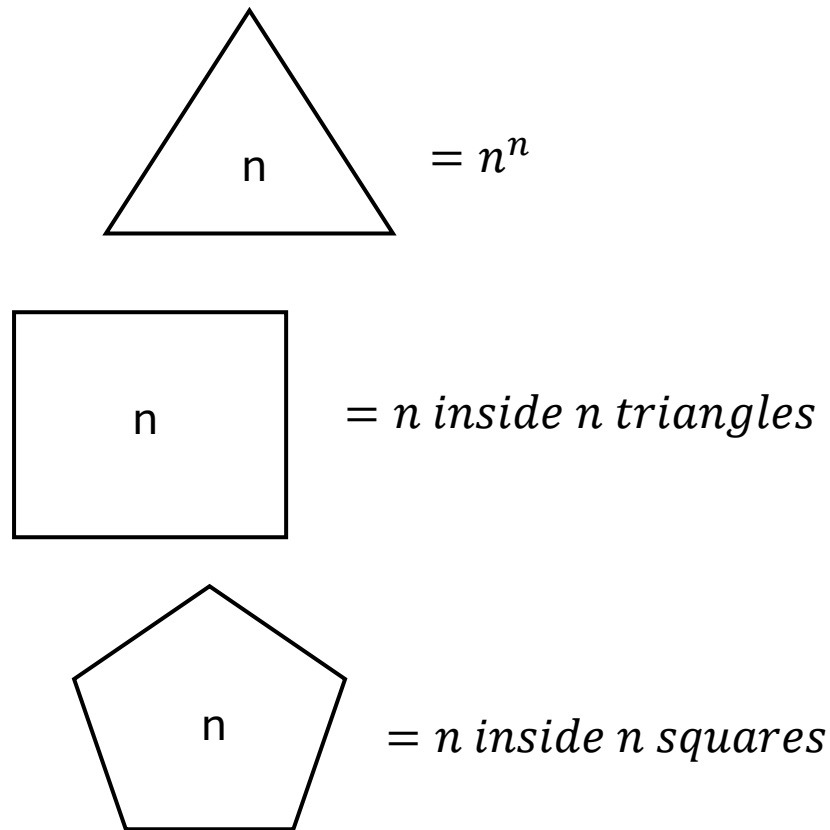


Figure 2 – Steinhaus-Moser Notation

For instance, $triangle(2) = 2^2 = 4$, and $triangle(3) = 3^3 = 27$. However, when we compute values for the $square(n)$ function, we see how quickly this escalates. For example,

$$square(2) = 2 \text{ inside } 2 \text{ triangles} = triangle(2)^{triangle(2)} = 4^4 = 256$$

We can also attempt to compute the following:

$$\begin{aligned}
 square(3) &= 3 \text{ inside } 3 \text{ triangles} \\
 &= triangle(triangle(triangle(3))) \\
 &= triangle(triangle(27)) \\
 &= triangle(27^{27}) \\
 &= INCOMPUTABLE \text{ (approx. } 10^{10^{40}})
 \end{aligned}$$

We have already reached numbers well beyond a googol and not far off a googolplex, and to many readers, this size will already be meaninglessly large. However, we are yet to reach a Mega.

According to Steinhaus and Moser, Mega is defined as $pentagon(2)$. Again, we can attempt to compute this.

$$\begin{aligned} pentagon(2) &= 2 \text{ inside } 2 \text{ squares} \\ &= square(square(2)) \\ &= square(256) \\ &= 256 \text{ inside } 256 \text{ triangles} \end{aligned}$$

This is astoundingly big, but Leo Moser decided to go even bigger.

The process of defining polygons could have been continued indefinitely, with Moser defining the following formula,

$$(S + 1)gon(n) = n \text{ inside } n (S) - gons$$

where an S -gon represents a polygon with S sides.

This is where we reach a Moser, defined as $megagon(2)$, which is 2 inside a polygon with Mega sides. We can also define a Super Moser as $mosergon(2)$, an astronomical number (Numberphile, The Hyper Moser (and other Mega Numbers) - Numberphile, 2025). It's incredible to think that these figures are just the start of this insane journey, as we can create even larger numbers.

Graham's Number

In order to explain this next gargantuan number, we must first introduce arrow notation. Let's start by doing $3 \uparrow 3$. We can rewrite this as $3^3 = 27$. Next, we will do the following:

$$\begin{aligned} 3 \uparrow \uparrow 3 &= 3 \uparrow (3 \uparrow 3) \\ &= 3 \uparrow 27 \\ &\approx 7.6 \text{ trillion} \end{aligned}$$

We can see that two arrows represents a power tower of 3s that is 3 numbers high. We now introduce three arrows, as below.

$$\begin{aligned} 3 \uparrow \uparrow \uparrow 3 &= 3 \uparrow \uparrow (3 \uparrow \uparrow 3) \\ &= 3 \uparrow \uparrow 7.6 \text{ trillion} \\ &= \text{MASSIVE} \end{aligned}$$

To even start looking at Graham's number, we need to add a fourth arrow. In this way, we can begin constructing the sequence g_n , where $g_1 = 3 \uparrow \uparrow \uparrow 3$. Now, I pose to you a question. To define g_2 , would we increase the number of arrows or the base of the operations (i.e. the number we are using)? If you said increase the number of arrows, you would be correct. If you thought there would be five or six of them, you are way off. In fact, there would now be g_1 of them. Algebraically, we can define the g_n sequence as:

$$g_{n+1} = 3 \uparrow (g_n \text{ arrows}) 3$$

We can now keep increasing in value, and eventually, when we reach g_{64} , we hit Graham's Number (Numberphile, Graham's Number - Numberphile, 2012). I'm sure you can see that this number is massive, but it's impossible to even describe the scale of this number. Despite all this, the number has been used constructively. The number was created by American mathematician Ron Graham in the 1970s as a solution to a combinatorics problem related to Ramsey Theory. Suppose you have a square, a two-dimensional (2D). You take the square and you draw a line between every vertex (corner). You also give each line a colour, red or blue, as shown in Figure 3.

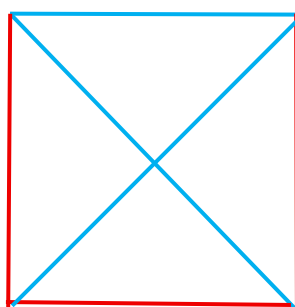


Figure 3 – A surprisingly constructive use of Graham's Number

Now suppose you take a cube, which is 3D, and you draw a line segment (red or blue) between every vertex. There would be 28 of these lines. You could repeat this process for a shape with greater and greater dimensions.

The main aim is to avoid any 2D face on the shape where all six lines are the same colour, and the number of dimensions where this is avoidable is between 13 and Graham's Number (Numberphile, What is Graham's Number? (feat Ron Graham), 2014) – a seemingly vast range, but tiny in the context of infinity.

The TREE Function

For our next number, we have to play a game of trees. To play this game, you take a number, n , of different seeds, and you draw trees with x number of seeds.

The rules of the game are as follows. On turn x , you draw a tree with up to x number of seeds (fewer seeds are allowed), which can be of the same or different types. An example is shown in Figure 4, using a red, green and blue seed.

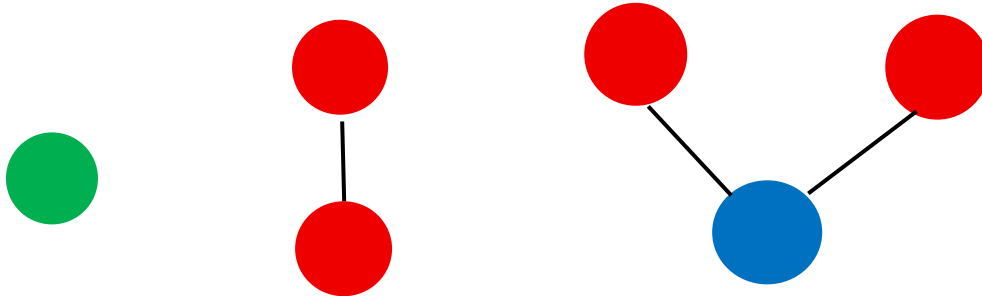


Figure 4 – A simple game of trees

You keep drawing trees until you find that a tree could have been ‘contained’ (the mathematical term is ‘inf-embeddable’) in an earlier tree, meaning that several of the same seeds preserve the same nearest common ancestor. For instance, the trees in Figure 5 would not be permitted as several seeds (those in boxes) share the same nearest common ancestor.

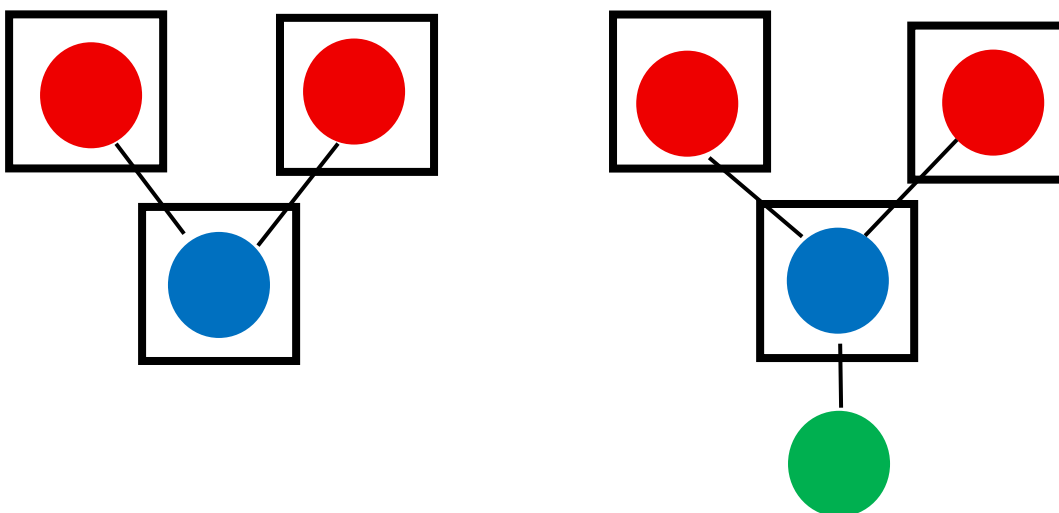


Figure 5 – The meaning of ‘contained’ (‘inf-embeddable’)

Now, we are ready to introduce the function $TREE(n)$, which represents the longest possible game that can be played in number of turns, x , and where n represents the

number of seeds. First, we will look at $TREE(1)$, using only a red seed. Shown in Figure 6 is the only possibly game which can be played.

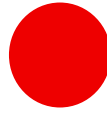


Figure 6 – A representation of $TREE(1)$

If another tree is drawn, it would contain within it this tree. Thus, only this tree can be drawn, and we can conclude that $TREE(1) = 1$.

We now look at $TREE(2)$, using a red and green seed. Again, the longest possible game is shown in Figure 7.

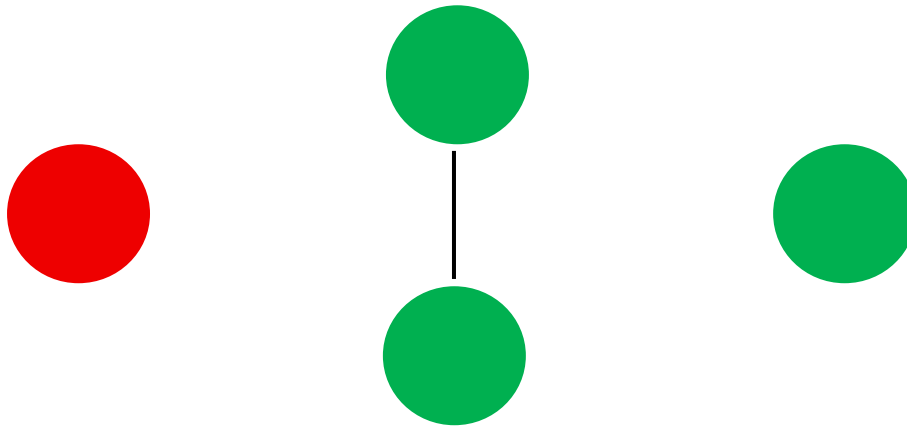


Figure 7 – A representation of $TREE(2)$

Any other tree would contain either a red or a green seed, therefore meaning that the longest possible game consists of 3 turns. We can say that $TREE(2) = 3$.

To model $TREE(3)$, we will use a red, green and blue seed. The start of a possible game is shown earlier, in Figure 4. It wouldn't be possible to show the full longest game, or even describe how long it would be. The number of turns is orders of magnitude greater than Graham's Number. In fact, this number is so enormous that nobody has been able to work out any of its last digits. It is so much larger as you wouldn't be able to re-use the seed used in the first tree again in any other tree. Having two more seeds available gives the player far more possible trees to draw (Numberphile, The Enormous $TREE(3)$ - Numberphile, 2017).

This number is related to Kruskal's Tree Theorem, which states that, for any set of finite trees, one tree is 'contained' within another tree and the game has to end. An interesting

fact about these numbers is that, using regular arithmetic, $TREE(n)$ can be proved to be finite for any specific integer n , but not for all n . The general proof that $TREE(n)$ is finite for all n can only be carried out using ordinals and other arithmetic related to infinity (Numberphile, TREE(3) (extra footage) - Numberphile, 2017).

Conclusion

Here, we have journeyed through some of the biggest numbers ever conjured up, all of which are incomprehensible to the human mind. At the start, I asked how big we can really go. You might ask why I haven't mentioned infinity. Infinity is not a number. Instead, it is a concept – the concept that numbers can continue indefinitely and that you can always add one. So, in answer to my question, we can go as big as we like.

Bibliography

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