A generalization of the problem of a particle sliding on a circular igloo to the case of an elliptical igloo geometry

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1 The circular igloo

The problem consists of a particle of mass m starting from rest at the top of a circular igloo, where there is no friction. The objective is to determine the angle at which it loses contact with the surface, measured with respect to the vertical. For this case, let r denote the radius (constant) of the igloo and consider the coordinates $A(0, r)$ and $B(r, 0)$

We will solve the problem using the principle of conservation of mechanical energy and concepts from particle dynamics.

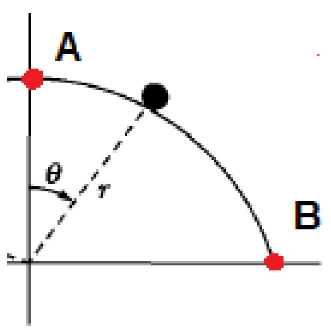

Figure 1: Elipse en en el primer cuadrante

In this case we kwnow that there are only two forces acting, whose are: The normal and the weight of the particle. If we consider the radial axis whe have $(r = \rho)$:

$$mg\cos(\theta) - N = \frac{mv^2}{\rho} \tag{1.1}$$

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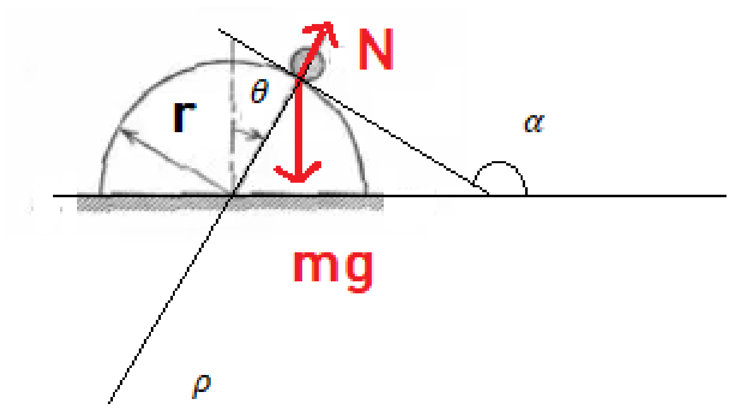

Figure 2: Particle sliding

where the right term correspond to the centripetal acceleration and g is the standart acceleration of the gravity. Applying the principle of conservation of energy at point A, where the energy is purely potential, and at the point where the normal force vanishes, we obtain the following equations:

$$N = 0 \tag{1.2}$$

$$mgr = mgr\cos(\theta) + \frac{1}{2}mv^2 \tag{1.3}$$

Solving for θ at 1.1, 1.2 and 1.3 finally we have

$$\theta = \cos^{-1}\left(\frac{2}{3}\right) \tag{1.4}$$

2 The eliptic igloo

Now we consider the eliptic (typlicar semi-axes a and b)igloo, according to the figure, the particle starts at the point $A(0, b)$

In this case the radius is not constan,we can parametrize the elipsis as following

$$\gamma(\theta) = (a\sin(\theta), b\cos(\theta)) \tag{2.1}$$

Let be P the point where the Normal vanishes, this point has cordinates $(a\sin(\theta), b\cos(\theta))$, now we van see that the slope at this point is obtained by differenciating the standar equation of the ellipsis with respect to x

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Then we have

$$\frac{dy}{dx} = -\frac{b^2 x}{a^2 y}$$

But, according to 2.1 finally we have

$$\tan(\alpha) = \frac{dy}{dx} = -\frac{b}{a}\tan(\theta) \tag{2.2}$$

Subsequently

$$\cos(\alpha) = \frac{a}{\sqrt{a^2 + b^2\tan^2\theta}}$$

We can also obtain the radius of curvature at this point using standard tools from vector calculus.

$$\rho = \frac{||\gamma||^3}{||\gamma' \times \gamma''||}$$

Then we have

$$\rho = \frac{(a^2\cos^2(\theta) + b^2\sin^2(\theta))^{(3/2)}}{ab} \tag{2.3}$$

Similary if we aply the principle of conservation at this case we obtain the following

$$mgb = mgr\cos(\theta) + \frac{1}{2}mv^2 \tag{2.4}$$

Reecall 1.1, and the similar triangles

$$v^2 = \rho g\ cos(\alpha)$$

And we consider the distance between the origin and P

$$r = \sqrt{a^2\sin^2\theta + b^2\cos^2\theta}$$

Putting all in 2.4 we have the relation between θ and the parameters a and b

$$mgb = mg\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}\cos(\theta) + \frac{1}{2}\frac{(a^2\cos^2(\theta) + b^2\sin^2(\theta))^{(3/2)}}{ab}\frac{mag}{\sqrt{a^2 + b^2\tan^2\theta}}$$

Simplifying

$$\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}\cos(\theta) + \frac{1}{2}\frac{(a^2\cos^2(\theta) + b^2\sin^2(\theta))^{(3/2)}}{b}\frac{1}{\sqrt{a^2 + b^2\tan^2\theta}} - b = 0$$

$$2b\cos(\theta)\sqrt{a^2\sin^2\theta + b^2\cos^2\theta} + \cos(\theta)(a^2\cos^2(\theta) + b^2\sin^2(\theta)) - 2b^2 = 0$$

If we make $a = b$ we clearly obtain 1.4.

Now we solve for θ using the change of variable $X^2 = \cos^2(\theta) \Rightarrow sin^2\theta = 1 - X^2$ we get the following equations after some algebra

$$(a^2b^2)^2X^4 + 2b^2(a^2b^2)X^2 + b^4a^2b^2 = 0$$

Solving the bicuadratic equation we have for $a > b$

$$\theta = \cos^{-1}\left(\frac{b}{\sqrt{a^2 - b^2}}\right)$$

Clearly, this formula is not valid for $a = b$, as it becomes singular. This is due to the fact that, during the solution of the biquadratic equation, we divide by the factor $a^2 - b^2$, which vanishes when $a = b$.

3 The Lagrangian approach

"Lagrangian mechanics allows us to solve problems without explicitly resorting to the concept of force, which is a vector quantity whose manipulation in problem-solving is often more complicated than working with scalar quantities, primarily energy. However, there are problems in which it is necessary to determine the nature of the forces involved.

In such cases, Lagrangian mechanics introduces the concept of non-holonomic constraint forces, which are associated with constraint functions f. These functions are expressed in terms of the positions q_i and velocities $\dot{q}_i$ of the particle or particles involved in the system under consideration. That is, such constraint functions have the general form

$$f(q_i, \dot{q}_i) = 0$$

By incorporating these constraint functions, the problem becomes restricted to certain conditions, which cause the Euler–Lagrange equations to take the following form

$$\sum_{i=1}^{n} \frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) + \sum_{k=1}^{m} \lambda_k \frac{\partial f_k}{\partial q_j}$$

With $i = 1, 2, ..., n$ number of degree of freedom and $k = 1, ..., m$ number of constaints. λ_k are the Lagrange multipliers also know as constraint forces

$$Q_i = \sum_{k=1}^{m} \lambda_k \frac{\partial f_k}{\partial q_j}$$

Initial conditions of the problem

i) $x(0) = y(\pi/2) = 0$

ii) $y(0) = b, \quad x(\pi/2) = a$

iii) $\dfrac{d^2y}{dx^2} < 0 \quad (0 < x < a)$

The constraint is given by

$$f(r,\theta) = r - \sqrt{a^2\sin^2\theta + b^2\cos^2\theta} = 0 \tag{3.1}$$

The Lagrangian is given by

$$L = T - U \tag{3.2}$$

Where:
The kinetic energy in polar coordinates is given by $T = \frac{1}{2}(\dot{r}^2 + r^2\dot{\theta}^2)$
Similary the potential energy $U = mgr\cos\theta$

Then the Lagrangian is

$$L(r,\theta,\dot{r},\dot{\theta}) = \frac{1}{2}(\dot{r}^2 + r^2\dot{\theta}^2) - mgr\cos\theta$$

The Euler Lagrange Equations are

$$\frac{\partial L}{\partial \theta} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} + \lambda\frac{\partial f}{\partial \theta} = 0$$
$$\frac{\partial L}{\partial r} - \frac{d}{dt}\frac{\partial L}{\partial \dot{r}} + \lambda\frac{\partial f}{\partial r} = 0$$

Making the partial fractions we have

$$mr\dot{\theta}^2 - mg\cos\theta - m\ddot{r} + \lambda = 0 \tag{3.3}$$

$$mgr\sin\theta - mr^2\ddot{\theta} - 2mr\dot{r}\dot{\theta} + \lambda\frac{\sin\theta\cos\theta(a^2-b^2)}{r} = 0 \tag{3.4}$$

Computing $\dot{r}$ y $\ddot{r}$ we have

$$\dot{r} = \frac{\dot{\theta}\sin\theta\cos\theta(a^2-b^2)}{\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}} \tag{3.5}$$

$$\ddot{r} = (a^2-b^2)\frac{\dot{\theta}^2(b^2\cos^4\theta - a^2\sin^4\theta) + \ddot{\theta}\sin\theta\cos\theta(a^2\sin^2\theta + b^2\cos^2\theta)}{(a^2\sin^2\theta + b^2\cos^2\theta)^{\frac{3}{2}}} \tag{3.6}$$

Finally we have

$$m\dot{\theta}^2\sqrt{a^2\sin^2\theta + b^2\cos^2\theta} - mg\cos\theta - \cdots$$

$$\cdots - m(a^2-b^2)\frac{\dot{\theta}^2(b^2\cos^4\theta - a^2\sin^4\theta) + \ddot{\theta}\sin\theta\cos\theta(a^2\sin^2\theta + b^2\cos^2\theta)}{(a^2\sin^2\theta + b^2\cos^2\theta)^{\frac{3}{2}}} + \lambda = 0$$

$$mg\sin\theta - m\ddot{\theta}\sqrt{a^2\sin^2\theta + b^2\cos^2\theta} - 2m\frac{\dot{\theta}^2\sin\theta\cos\theta(a^2-b^2)}{\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}} + \lambda\frac{\sin\theta\cos\theta(a^2-b^2)}{a^2\sin^2\theta + b^2\cos^2\theta} = 0$$

4 Conclusion

Both Newtonian mechanics and Lagrangian formalism provide powerful frameworks for solving problems in physics, yet they differ fundamentally in their approach and conceptual structure. Newtonian mechanics is grounded in the direct application of forces and vector equations of motion, offering an intuitive and physically transparent description of dynamics. It is particularly effective in problems where forces are explicitly known and easily characterized.

In contrast, the Lagrangian formalism shifts the focus from forces to energy, working with scalar quantities such as kinetic and potential energy. This approach often simplifies the mathematical treatment of complex systems, especially those involving constraints or generalized coordinates. Moreover, it provides a more systematic and elegant framework for dealing with constrained motion and for extending classical mechanics into more advanced theories, including field theory and modern theoretical physics.

Ultimately, while Newtonian mechanics offers direct physical insight, Lagrangian mechanics provides greater flexibility and mathematical efficiency. Both formulations are fundamentally equivalent in their physical predictions, and their usefulness depends on the nature of the problem under consideration.

It can be seen that the Lagrangian formalism is somewhat more elaborate for this class of problems. In both formulations, a general expression relating the angle to the parameters a and b is obtained. However, in the Lagrangian framework, the formal structure of the theory leads to a system of two equations rather than a single one.

P.S. If we wish to further complicate the formalism, we could adopt the Hamiltonian approach—but that is another story. In that case, we would obtain four equations by performing a Legendre transformation.