Banach, Tarski, and Theseus

1913 words excluding captions and this title.

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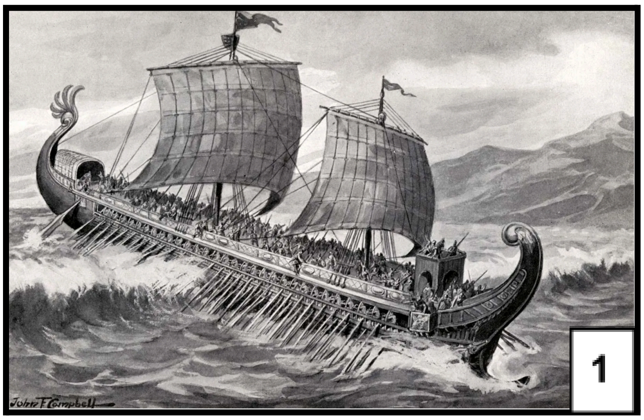
Picture this fictional scene – **Theseus**, divine Greek mythological hero, sits pensively on his throne in Athens, obsessed with a question of cosmic importance:

If every piece of my ship were to be replaced one by one,
would its identity change?



This question has captivated philosophers (perhaps shipwrights too, who knows?) for millennia. Mathematics has often been applied to try and answer questions from Greek mythology, such as Zeno's paradoxes or the Hydra game. However, this essay will *not* resolve Theseus's burning question- quite the contrary:

In the domain of theoretical mathematics,
Theseus may not have a defined ship at all.

With the help of two demigods of our own time, **Stefan Banach** [2] and **Alfred Tarski** [3], let us explore this statement, and the **Banach-Tarski paradox**.

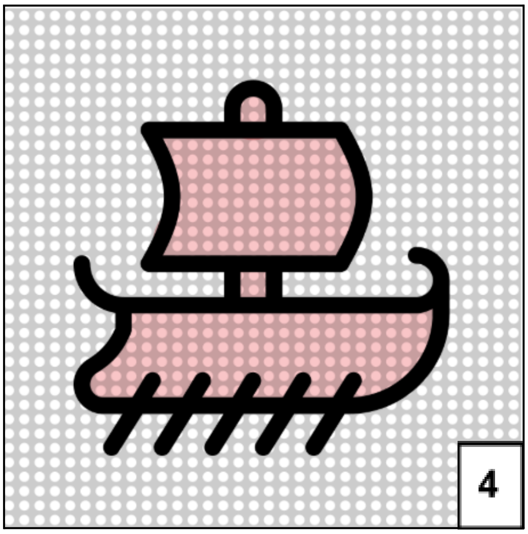
1.1 The Ship of Theseus

The philosophical thought followed by Theseus and other philosophers is this:

If identity depends both on materials and shape,
any replacement changes the ship.
If only shape is required, changing materials has no effect.

So far, *smooth sailing* (pun intended), but how *does* shape define identity?

Mathematically, we can define the shape of an object as the subset of all points within 3D space which are considered part of the object:

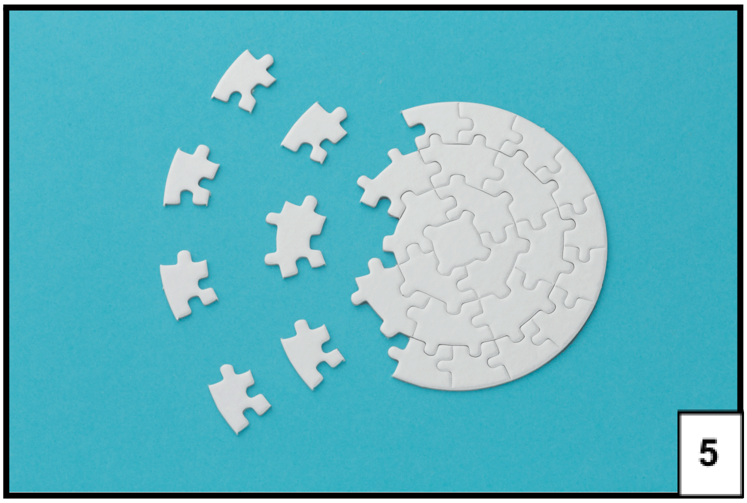


[2D example: the red points are a subset of all points.]

Any 3D object ***A*** can be decomposed into disjoint constituent sets ***B***, ***C***, *etc*, the union of which reforms ***A***.

$$A = B \cup C \cup \ldots,\ B \cap C \cap \ldots = \varnothing$$

Thus, subsets (pieces) of a whole (the ship) "carry" its identity, if their combination *always* reconstructs the object.



[Visual analogy of pieces carrying the identity of the whole through deconstruction.]

As such, the Ship of Theseus can be dismantled and reconstructed (instantly or, as the story goes, over a period of time) back into the same shape, preserving its identity.

Right?

1.2 The *Mathematical* Ship of Theseus

Enter the **Banach – Tarski paradox**, the *anchor* of this essay. It states:

Given a solid ball in 3D space, there exists a decomposition of the ball into a finite number of disjoint subsets that can be put back together in a different way to yield two identical copies of the original ball.

Simply: Objects can be duplicated simply through **partition** and **reassembly**, *without addition of external volume.*

This implies that once dismantled, pieces can be reconstructed into more than just the original object, demonstrating that its shape does *not* preserve its identity.

Mathematically, the Ship of Theseus has no stable identity from the beginning, if we define identity solely based on shape.

But how does this work? Let us follow **Banach** and **Tarski**, dressed in white robes and brought to mythological Greece, as they duplicate a sphere before Theseus's bewildered eyes.

2 Preliminaries

“We cannot simply duplicate *any* real sphere,” Tarski tells Theseus, “two conditions need to be assumed.”

2.1 Condition A: **The Axiom of Choice:**

$$\forall F((\forall A \in F, A \neq \emptyset) \Rightarrow \exists f : F \to \bigcup F \text{ such that } \forall A \in F, f(A) \in A)$$

An intimidating expression, which Theseus bravely faces.

$$\forall F((\forall A \in F, A \neq \emptyset)$$

For every collection of sets F with non-empty subsets…

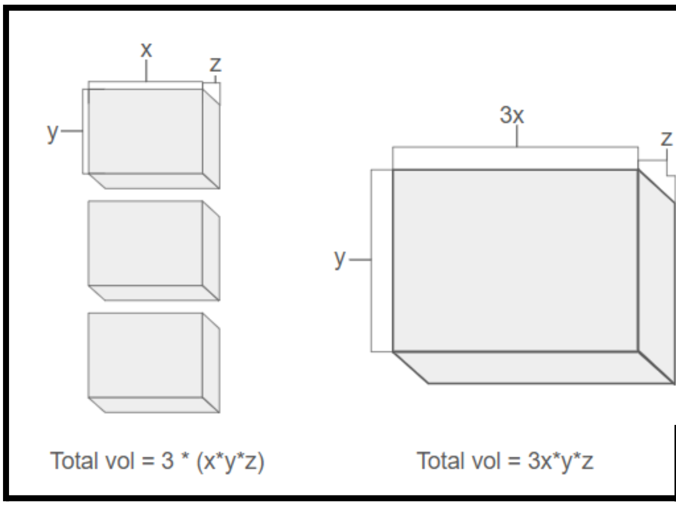
$$\exists f : F \to \bigcup F \text{ such that } \forall A \in F, f(A) \in A)$$

…there exists a function which returns an element of its input subset.

This axiom enables the arbitrary selection of elements from *infinitely* many subsets, without a specific rule. This is impossible in reality, so we thank the imaginary nature of our protagonists .

2.2 Condition B: **Infinitesimal division:**

Intuitively, the **volume** of a 3D shape equals the volume of its components [6], under **isometric** (rigid) transformations.



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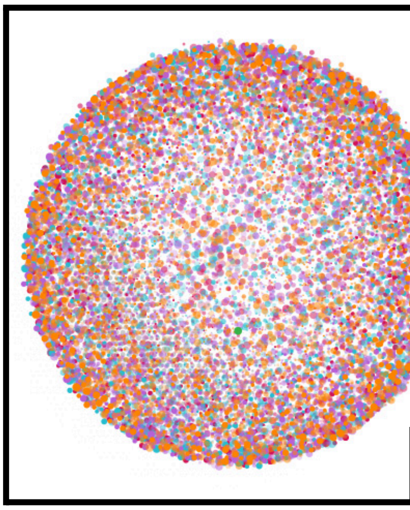


7

"The rule- *no adding external volume* - means we must use only these isometries," Banach says (in Polish, which Hermes translates for Theseus).

Partitioning shapes into non-measurable sets of uncountably infinite points [7], *enables* these isometries to increase total volume. **Real** objects cannot be infinitely divided, so we assume the **theoretical** sphere can.

Unfortunately, Theseus is no Daedalus, so Banach must clarify two more points.

2.3 **Free-groups:**

These are sets built from generators and an operation. Generators are simply "building blocks" for group elements, and operations combine them.

A free-group ***F*** is *only* constrained by the 4 rules of groups:

a) **Closure**:

$$x, y \in F,\ x*y \in F$$

Elements combined through the operation are elements of the group.

b) **Identity:**

$$y*e = e*y = y$$

There is a neutral identity element *e*.

c) **Inverses:**

$$x^{-1}*x = e$$

Every generator ***X*** has an inverse which it cancels with.

d) **Associativity:**

$$(a*b)*c\ =\ a*(b*c)$$

Combination allows any grouping.

Theseus: "Show me."
Tarski: "*Shore* thing. For generator set ***G*** and operation *****, the free group would be:"

$$G = \{A, B\}$$

$$F = \{A, B, A^{-1}, B^{-1}, A*A, A*B, ..., A*B^{-1}*A^{-1}*B*B*A, ...\}$$

Free-groups are used in the Banach-Tarski paradox as their elements have *no additional relations*, so different elements will not collapse into one.

Example:

$$ab \neq ba$$

2.4 **Arccos (⅓)**

A point on a circumference returns to its starting position if it rotates in steps of a **rational** multiple of 360°.
Example:

$$\frac{1}{2} \times 360° = 2 \textit{ rotations before returning}$$

However, steps of an **irrational** multiple (**arccos (⅓)**) never repeat the same position.

Proof by contradiction:

Assume;

$$x \times n = m \times 360 \quad where \quad m, n \in \mathbb{Z}^{+}$$

Ie: ***n*** rotations by ***X*** (irrational multiple) *do* cover the same distance as ***m*** complete rotations.

Then;

$$x = \frac{m \times 360}{n}, \textit{ and thus } x \in \mathbb{Q}$$

Contradiction, since ***X*** is irrational. Hence ***X***-steps never repeat a point.
Since **arccos (⅓)** is irrational, it behaves in this way as well.

3 Duplication

3.1 *Defining parts of the sphere*

Banach begins: "Imagine two perpendicular axes intersecting at the sphere's center."

He then defines generators **{*U, R*}**. For any point on the surface of the sphere:
U = upwards rotation (towards the North pole)
R = rightwards rotation (towards the East pole).
Inverses U^{-1} and R^{-1} are labelled ***D*** (down) and ***L*** (left) respectively.

Each rotation is **arccos(⅓)** around the center.

"Why?" questions the all-curious Theseus.

"Trivial." replies a divine owl on a nearby olive tree.

Banach elaborates: "It ensures no sequence of rotations maps a point back to itself, unless it backtracks immediately, such as ***UD*** or ***RL***."

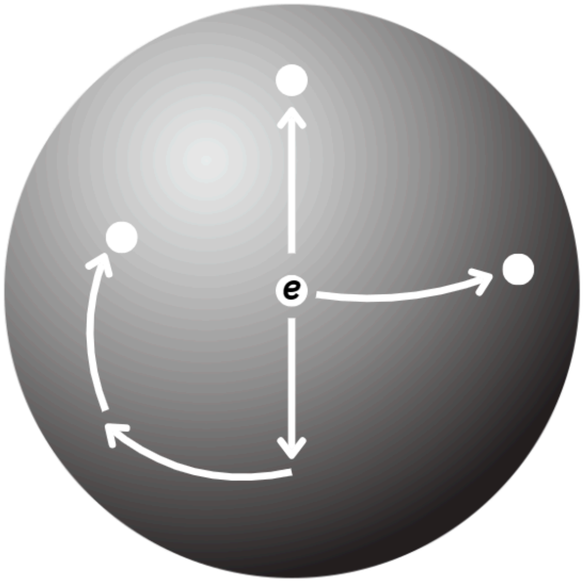
From these **4** possible rotations, they construct the free-group F_2 of all possible sequences:

$$\{L, LU, UUR, RULD, \ldots\}$$

Every sequence is *reduced*- any backtrack ***LR***, ***UD***, etc, is removed.

"*Aboat* time," Tarski says once finally done with meticulously writing down this infinite number of sequences.

He then defines group G_1, with an arbitrary surface point as identity element ***e***. With each F_2 sequence he maps point ***e*** to another distinct point, which he adds to G_1.



[Example of ***e*** and 3 other points]

F_2 is countably infinite, so ***G_1*** is as well. However there are uncountably many surface points, ergo uncountably many unmapped points remain.

Sighing, Tarski begins painstakingly selecting uncountably many starting points from the remaining surface, building:

$$G_2, G_3, G_4, \ldots G_n$$

Note that without the **Axiom of Choice**, choosing infinite points ***e*** to define infinite sets ***G_n*** would be impossible.

Eventually Banach maps all points, with the exception of **poles**. These can be mapped by multiple different sequences and are therefore ambiguous:

$$\ldots U, \ldots UU, \ldots UUU, \text{etc}$$

[Example: three different sequences mapping to the North pole]

Let poles have their own set, ***S(P)***, instead.
Likewise, assign starting points ***e*** to ***S(e)***.

“Lovely, but I still see only one sphere.” Theseus *stern*ly remarks.
Unbeknown to him, the veracity of that statement will be short lived.

3.2 *Decomposition*

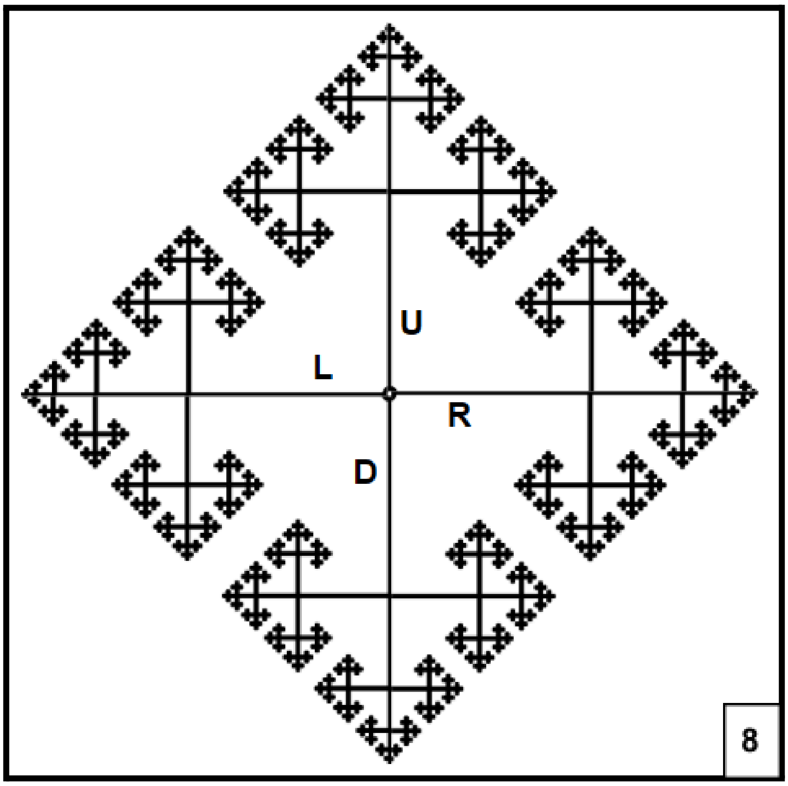
Our heroes now deconstruct the sphere surface into **6** sets:

1. Poles, ***S(P)***
2. Starting points, ***S(e)***

3-6. Rotation sequences starting with ***U, D, L*** and ***R*** respectively.

$$F_2 = S(e) \cup S(U) \cup S(D) \cup S(R) \cup S(L) \cup S(P)$$

Every point in F_2 is extended radially inwards to fill the circle into a **3D** shape. The very center is removed separately, *and will be used later*.



[Free-group F_2 can be visually represented by the above Cayley tree. Each intersection is an element of... $F_2 - (S(e) \cup S(P))$ ...represents a sequence of rotations, based on the orientation of the lines which map it from the center].

3.3 *Rearrangement*

[The following steps can be done starting with any set of:]

$$S(U),\ S(D),\ S(R),\ S(L)$$

Tarski chooses to start with ***S(R)*** - all points reached by sequences starting with a **rightwards** rotation. Its elements are:

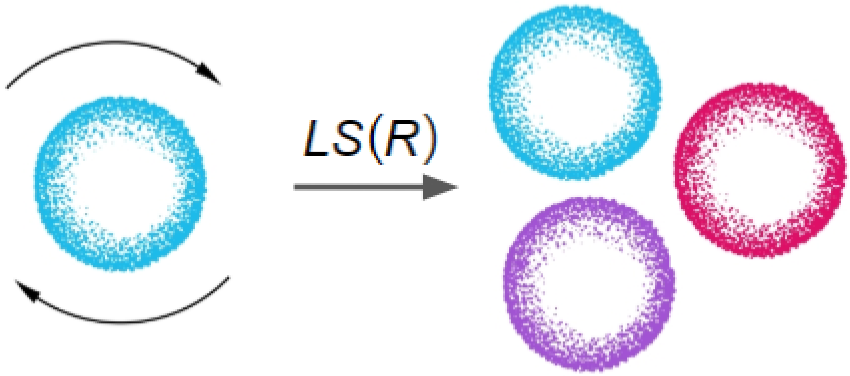
$\{R,$

$RR, RU, RD,$

$RRR, RRU, RRD, RUR, RUU, RUL, RDR, RDD, RDL,$

$\ldots\}$

Tarski rotates the sphere to the **left**, thus imposing an additional **leftwards** rotation to each sequence:



9

The gods of Olympus quiver before the power of mathematics as *2 more sets appear*.

“What?” asks Theseus, his confusion greater than ever.
“Shush,” we whisper, eager to hear the explanation (right?).

Rotating the whole set applies an initial **leftwards** rotation to each element, *reducing* each element and thus removing the initial ***R*** of each one. The set of all sequences starting with ***R*** becomes the set of all sequences starting with ***R, U*** or ***D*** (not ***L***, as any sequence starting with ***RL*** would simply reduce):

$\{e,$

R, U, D

$RR, RU, RD, UR, UL, UU, DR, DL, DD,$

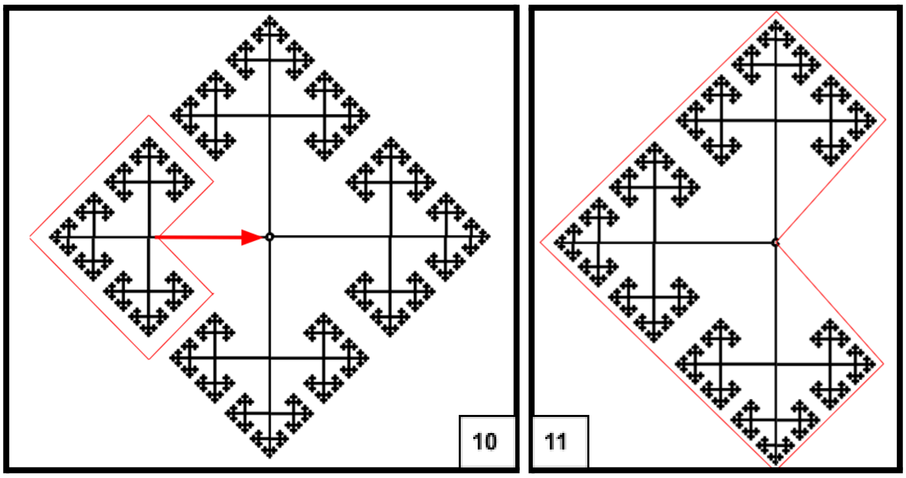
$\ldots\}$

$x \in S(R),\ x = RUDLD$

$LS(R) \Rightarrow Lx = LRUDLD = UDLD$

$UDLD \in S(U)$

[Left: ***LS(R)***, or ***S(R)*** U ***S(U)*** U ***S(D)*** U ***S(e)***. Right: example ***S(U)*** element formed in ***LS(R)***]



[The Cayley tree illustrates this step visually. We can see that eliminating the first step of one of the 4 branches (sets) creates ¾ of the branches.]

Note that the sequence ***R*** completely reduces to the identity element ***e***, so ***LS(R)*** contains ***S(e)*** as well.

Banach then adds the ***S(L)*** set, taken from the original sphere, to ***LS(R)***, obtaining:

$$LS(R) \cup S(L) = S(R) \cup S(L) \cup S(U) \cup S(D) \cup S(e) = F_2 - S(P)$$

The procedure above is repeated with sets ***S(U)*** and ***S(D)***, a downwards rotation being applied to form ***DS(U)***, and so on. *Two copies of the ball now exist*, each missing its **poles** and **center**. From the *original* sphere, the set of all poles, starting points and the very center are left over.

On the edge of his throne, accompanied by a pantheon of curious gods, Theseus eagerly watches the mathematicians work their logic (or magic, for those who do not understand what is going on).

3.4 *Poles, starting points, and center points.*

Tarski easily gets rid of the extra starting points set ***S(e)***, *countably* infinite, by adding it to either *uncountably* infinite sphere, since:

$$K + \aleph_0 = K$$

[Adding countable infinity to uncountable infinity amount does not change cardinality ("size").]

S(e) essentially vanishes into the sphere.

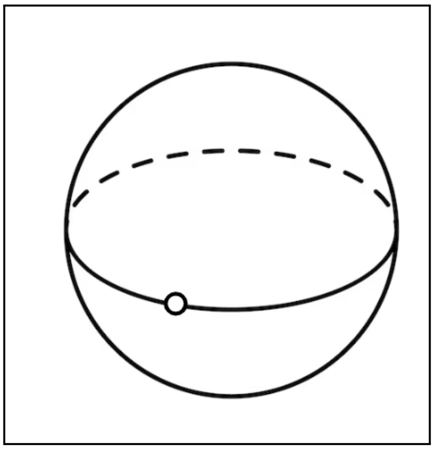
"As for poles," the two Poles explain, "we have *one* set for *two* incomplete spheres."

They add the set of poles to one of the two spheres, *along with the center piece*, reconstructing it into F_2 completely.

Theseus questions: "Where will we find the second set of poles and center point needed to fill the holes of the other sphere?"

"No need," the duo replies.

There are *countably* many missing pole-points in the second sphere, as F_2 contains countably many elements, and each is responsible for *up to **2*** missing pole-points.
Each missing point sits on some arbitrary circumference:

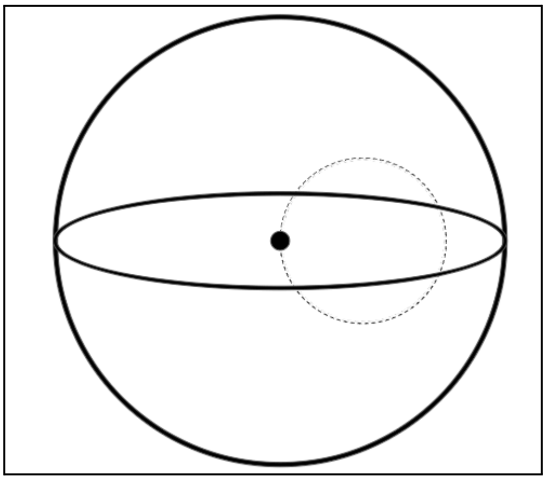


As shown earlier, steps of **arccos(⅓)** along a circumference never repeat a point. Thus, this circumference can be expressed as an infinite set of points **arccos(⅓)** apart, with one hole:

$$Circumference = \{hole, P_1, P_2, P_3, \ldots\}$$

We can therefore rotate the circumference such that each point is shifted to the left. $\boldsymbol{P_1}$ fills the hole, $\boldsymbol{P_2}$ fills the hole left by $\boldsymbol{P_1}$, and so on. Since there are infinitely many points, the hole disappears. In this way, the sphere's surface is made whole.

The center point can be filled in the same way, since it can be considered to lie on some arbitrary internal circumference.



With its center point added, the second sphere, too, becomes a perfect reconstruction of $\mathbf{F_2}$.

" *Voilà* " Banach and Tarski exclaim in unison…

…each holding a perfect copy of the original sphere in their hand.

"Holy ship!" says Theseus.

4 Conclusion

"You *sea*? Your logic is incomplete," Tarski tells the solemn King, "you ask if your ship will be the same after changing the materials, but don't even know *what* your ship is in the first place."

To which Theseus replies: "*Oar* you kidding me?"

Ultimately, the question of **Ship of Theseus** assumes that the ship's *shape* is sufficient to define its *identity*. This may be true in *physical* reality, where aforementioned theoretical conditions [**Axiom of Choice** and **infinite divisibility**] are inapplicable.
However, **Banach** and **Tarski** show us with their paradox that in *theoretical* mathematical environments under non-measurable decomposition, shapes can be partitioned and reassembled into more than their unique initial form.

Shape fails to preserve identity.

Therefore, the philosophical question fails to be analyzed theoretically, and is provably limited to realistic contexts. This also challenges whether not being theoretically valid is problematic. Perhaps the value of the philosophical question lies *not* in how mathematically provable it is, but how it *can* be answered when it cannot handle mathematics.

And so Theseus sits on his throne, eyeing his ship (s?) and wondering if his question is too broad, or the mathematics too specific. We leave the pensive hero to his reflections, and I thank you, the reader, for taking the time to follow me through the pages of this essay. I hope it made you learn something new, smile, or ideally, both.

Carpe diem. Seas the day.

Diagram references:

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[2] http://kielich.amu.edu.pl/Stefan_Banach/e-biography.html

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[4] Own image.

[5] https://www.freepik.com/free-photo/close-up-puzzle-background_22895087.htm#fromView=keyword&page=1&position=42&uuid=2710610f-42d8-44e7-9c1f-e00853cc40cb&query=Puzzle+pieces

[6] Own image.

[7] https://www.quantamagazine.org/how-a-mathematical-paradox-allows-infinite-cloning-20210826/

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[8, 10, 11] https://en.wikipedia.org/wiki/Cayley_graph