Not All Infinities Are Created Equal

An essay on Cantor’s theory of infinite sets

Here is something nobody tells you in school: infinity is not one thing. We all learn at some point that numbers go on forever — you can always add one more — and we nod along and assume that settles it. Infinity means endless, and endless is endless. What more is there to say?

As it turns out, quite a lot. In the nineteenth century, a mathematician named Georg Cantor asked a question that sounds almost childish: are some infinities bigger than others? The answer he found — rigorously, beautifully, and against fierce opposition — was yes. Not all infinities are the same size. Some are, in a precise mathematical sense, vastly larger than others. And the proof is something any careful reader can follow, no specialist training required.

What Does “Same Size” Actually Mean?

Before we can talk about different-sized infinities, we need to agree on what it even means for two collections to be the same size. For small numbers, this is trivial: you count them. But counting falls apart the moment you try to apply it to something infinite. You cannot count to infinity and check the total.

So mathematicians use a more elegant idea: **matching**. Two collections are the same size if you can pair every element of one with exactly one element of the other, leaving nothing unpaired on either side. Think of it as a dance — if every person on the floor finds exactly one partner and nobody is left standing alone, the two groups are equal in size. No counting needed.

This works perfectly for finite collections. Now watch what happens when we let things get infinite.

The Even Numbers Aren’t Smaller Than You Think

Take the natural numbers: 1, 2, 3, 4, 5, and so on. Now take just the even ones: 2, 4, 6, 8, 10… Your instinct almost certainly says the second list is smaller. It’s a subset — only every other number makes the cut. The full list of naturals includes all the odd ones too. Surely it must be bigger.

But try the matching. Pair 1 with 2. Pair 2 with 4. Pair 3 with 6. In general, pair every natural number n with the even number $2n$. Every natural number gets a partner. Every even number is spoken for. Nobody is left out.

By the matching definition, the two collections are exactly the same size. The evens — this apparent "half" of the naturals — are just as numerous as the naturals themselves. This is the first genuinely strange thing about infinity: a set can be the same size as a part of itself. For finite collections, that's impossible. For infinite ones, it's just Tuesday.

What About Fractions?

Fine, you might say. Even numbers and natural numbers, both infinite, same size. Odd trick, but perhaps everything infinite just ends up equal. What about fractions?

Between 0 and 1 alone, there are infinitely many fractions: 1/2, 1/3, 1/4, 1/5, then 2/3, 3/4, 3/5, 3/7… The rational numbers — all fractions of the form p/q — seem hopelessly denser than the integers. Between any two whole numbers, you can fit infinitely many fractions. Surely this is a larger infinity.

Cantor showed it is not. He devised a method of arranging all possible fractions in an infinite grid — rows for numerators, columns for denominators — and then tracing a zigzagging diagonal path through it. This path visits every single fraction, one by one, and pairs each with a natural number. The matching works perfectly.

The rational numbers are **countable**: the same infinite size as the natural numbers. Cantor gave this size a name — $\aleph_0$, aleph-null — and it is the smallest infinity that exists. The fractions, for all their apparent density, are no more numerous than 1, 2, 3, 4, 5…

Where It All Breaks Down

At this point, you might reasonably suspect that every infinite collection turns out countable — that aleph-null is the only infinity there is. This is exactly what Cantor disproved, and he did it with one of the most ingenious arguments in mathematical history.

Consider the real numbers: not just fractions, but everything on the continuous number line. Every decimal that goes on forever, whether in a repeating pattern or not — numbers like $\pi = 3.14159\ldots$ or $\sqrt{2} = 1.41421\ldots$, or a completely random infinite string of digits that never settles into any pattern. Can we match these with the natural numbers?

Suppose someone claims they can. They hand you a list: real number 1, real number 2, real number 3, supposedly containing every real number between 0 and 1 without exception. Cantor says: I can always find one you missed.

Here is how. Look at the first decimal digit of the first number on the list — change it. Look at the second decimal digit of the second number — change that too. Third digit of the third number. Fourth digit of the fourth. Keep going forever. You have now built a brand new real number, and it is guaranteed to be missing from the list — because it differs from entry one in position one, from entry two in position two, from entry three in position three, and so on down the line. No matter where you look for it, it isn't there.

This is Cantor's diagonal argument, and its conclusion is devastating: no list can contain all the real numbers. They cannot be matched with the natural numbers. They form a strictly **larger** infinity — something beyond aleph-null, something uncountably vast.

An Endless Tower

And here is where things become almost vertiginous. Cantor did not stop at two sizes of infinity. He proved that you can always build a larger infinity from any infinity you already have, by considering all possible subsets of that collection. From aleph-null, you get the size of the real numbers. From that, you get something larger still. The tower has no ceiling.

So when we say "infinity," we are not naming a single destination. We are gesturing at an entire hierarchy — a structured, never-ending sequence of distinct magnitudes, each one rigorously, provably larger than the last.

The Human Cost of an Abstract Idea

None of this was easy, and not only because the mathematics is subtle. Cantor's contemporaries largely despised his work. The mathematician Leopold Kronecker — influential, sharp-tongued, and deeply opposed to the whole idea of actual infinities — reportedly called Cantor a "corruptor of youth." The attacks were personal and sustained, and Cantor, who was not a robust man to begin with, suffered badly for them. He spent long stretches of his later life in psychiatric care.

He died in 1918 in a sanatorium, his ideas still contested in some quarters. Within a generation, the verdict had reversed entirely. David Hilbert, the dominant mathematician of the early twentieth century, put it simply: no one shall expel us from the paradise that Cantor has created. Today, Cantor's set theory is not a curiosity — it is the bedrock on which modern mathematics is built.

Why Any of This Matters

It would be reasonable to ask whether any of this escapes the philosopher's armchair. It does. Alan Turing, the father of computer science, used a close cousin of Cantor's diagonal argument to prove that certain problems are **undecidable** — not just hard, not just unsolved, but permanently beyond the reach of any algorithm that could ever be written. The reason is essentially the same: the space of possible programs is countable, while the space of problems is not. Some things will always slip through.

More broadly, Cantor’s work is a reminder of what mathematics actually is. It is not arithmetic. It is not the manipulation of symbols according to rules. It is the discipline of following an argument wherever logic leads, even when the destination seems impossible — and then proving, beyond any doubt, that you have arrived somewhere real.

The next time someone uses the word “infinity” as if it settles a question, it is worth remembering: the landscape is far stranger, and far richer, than the word alone suggests.