PREY-PREDATOR MODEL ADAPTATION

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1. Introduction

Imagine a world full of dragons and phoenixes that are at odds with each other in every way possible except for hideouts. Their diets are completely different and are based on how their digestive systems are set up. Phoenixes have a ruminant stomach that allows them to simply eat plants or other leafy greens. Whereas dragons have a monogastric stomach and their primary diet is meat. Now, these animals are not competitive when it comes to their habitats, but they are when it comes to other resources such as water.

This is where conservationist Bob comes in. Bob studies these animals' conservation needs using mathematics, specifically a prey-predator model, and provides his company, Math Major Inc., with information on how best to help dragons and phoenixes. Math Major Inc.'s work has been notable in protecting both animals by recovering or replacing their lost resources.

The information is relevant because we will use the prey-predator model to explore the dynamics of two species. Specifically, we will examine how their populations change in response to their resource needs and how the number of individuals in one species can influence those of another.

2. Background

In this world, there is a forest called Annabelle Forest and a desert called Alexander Desert, where dragons and phoenixes reside, respectively. This land is dwindling because of the opportunistic Fae residents' consumption and greed, thus a decrease in both species' populations, specifically the females. Since this occurred, Bob has been informing Math Major Inc. about actionable objectives on how best to help the dragons and phoenixes.

Some of the measures implemented are setting up queries within the government system, establishing animal rescues, and using mathematical models to determine how to preserve the most needed resource. The most essential resource is water, which is a stream that separates the forest and the desert.

Given the greed of the Fae population, water is a popular item for them to take or develop dams to resolve their own issues with a lack of water supply. In turn, it leads to the downfall of dragons and phoenixes as medical issues arise, such as dehydration and death. This is where Math Major Inc. and Bob come in to develop tasks that prevent more of this harm from occurring.

As a result, they use the prey-predator model to help explain to the government system why letting the populations of dragons and phoenixes diminish is harmful to the environment all around. For the discussion of the primary resource between both species is water, the model will be used to reflect more on Fae and them instead. Parameters will be reflected based on what I think is an accurate value for these endangered animals.

3. Model

3.1. **Prey-Predator Model.** The prey-predator model was constructed on the basic assumption of growth rates between two animal populations [8]. Simply, modeling helps to determine which population between the two species will flourish and/or decline due to the lack of water. There is a possibility that it will affect both of them simultaneously.

The model follows:

$$f(x) = \frac{dx}{dt} = \alpha x - \beta xy \tag{1}$$

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$$g(x) = \frac{dy}{dt} = -\gamma y + \delta xy \tag{2}$$

where α, β, γ, and δ are positive constants.

Below, Table 1 explains the application of each variable found within this model [1, 4, 2].

x	Prey's population
y	Predator's population
α	Prey's growth rate
β	Prey population proportional to predators population
γ	Predators' death rate
δ	Increase of both populations

Table 1. Function of Variables

3.2. **Solving the Prey-predator Model.** In equations (1) and (2), points (x, y) are identified to explain how the solution develops over time and yet remains independent of time. These points are denoted as equilibrium points [3].

Finding the Equilibrium Points. Let's start by equaling (1) and (2) to zero and solve for x and y. This will create two sets of equilibrium points.

$$x(\alpha - \beta y) = 0 \implies x = 0; y = \frac{\alpha}{\beta}$$

$$y(-\gamma + \delta x) = 0 \implies y = 0; x = \frac{\gamma}{\delta}$$

The equilibrium points are $P_1 = (0, 0)$ and $P_2 = (\frac{\alpha}{\beta}, \frac{\gamma}{\delta})$. As the prey-predator model is nonlinear, the Jacobian matrix is calculated to find the linear version of the model [3].

Calculating the Jacobian matrix and Finding the Eigenvalues. A Jacobian matrix is computed by taking the partial derivatives of (1) and (2), shown below:

$$J = \begin{pmatrix} \frac{df}{dx} & \frac{df}{dy} \\ \frac{dg}{dx} & \frac{dg}{dy} \end{pmatrix} = \begin{pmatrix} \alpha - \beta y & -\beta x \\ -\delta y & \gamma - \delta x \end{pmatrix}$$

Next, the equilibrium points are used to find the eigenvalues. To find the linear system of P_1 and P_2, insert the equilibrium points into the Jacobian matrix, as such:

$$J(P_1) = \begin{pmatrix} \alpha - \beta(0) & -\beta(0) \\ -\delta(0) & \gamma - \delta(0) \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ 0 & \gamma \end{pmatrix}$$

$$J(P_2) = \begin{pmatrix} \alpha - \beta(\frac{\alpha}{\beta}) & -\beta(\frac{\gamma}{\delta}) \\ -\delta(\frac{\alpha}{\beta}) & \gamma - \delta(\frac{\gamma}{\delta}) \end{pmatrix} = \begin{pmatrix} 0 & -\beta(\frac{\gamma}{\delta}) \\ -\delta(\frac{\alpha}{\beta}) & 0 \end{pmatrix}$$

For the eigenvalues, the determinant is calculated by det(J-) for $P_1 and P_2$.

Since we know the eigenvalues for P_1 and P_2, the stability can be found for each equilibrium point. The stability is important to understand since it provides information about the interaction between two species [3].

4. Discussion

As seen in Table 1, there are four parameters $(\alpha, \beta, \gamma, and \delta)$ that are used to find the eigenvalues (λ). The constant α is represented twice to symbolize both dragons and phoenixes. Between both analyses of the prey-predator model, γ will be consistent since it represents the predator's death population. This is set at 0.01, which signifies dividing the number of deaths per year by the total population.

4.1. **Analysis of Dragon's population.** α calculates the birth rates of the selected population and is depicted by dividing the number of live births by the total yearly population. For the dragon's birth population, there is a total of 2,778 individuals, where females make up 40 percent of the population. On average, about 60 percent of all eggs laid in a season survive, making a total of 2,335 hatchlings included in the rate calculation. This makes $\alpha = 0.84$. Since the birth rate is high, I set β and δ to be higher than normal, as seen with this model, which is 0.15.

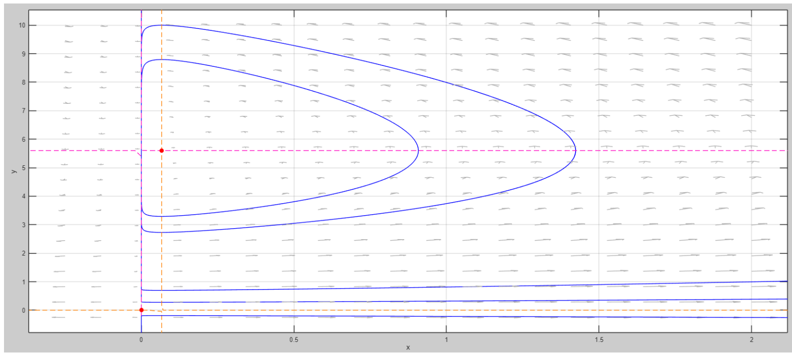

FIGURE 1. Phase Plane of the Dragon's Population [5]

In Figure 1, the blue trajectory lines represent the phase portrait of this system. There are two points, $P_1 and P_2$. P_1 is located at the origin, and the stability of this point is a saddle node. The analysis of P_2 is centered, meaning that the dragon's population will stay consistent since there is a larger number of individuals within their population. This can fluctuate based on how quickly their water sources are removed, which would cause a significant decrease in population values. Or another example of why a notable decrease would occur is if the Fae population started to exploit them by hunting or destroying their land.

4.2. **Analysis of Phoenix's population.** For the corresponding α, it is set at 0.2. The phoenix population is at 2,078, and the female population is 1,455. Phoenixes only lay one egg per mating season. Let's say that only 40 percent of the female inhabitants are reproducing and only 50 percent of those laid survive. So, the value that survives is 416 hatchlings. Taking 416 and dividing it by 2,078 will obtain 0.2. Since the birth rate for Phoenixes is low compared to the dragons, β and δ are relatively smaller as well. They are set at 0.05.

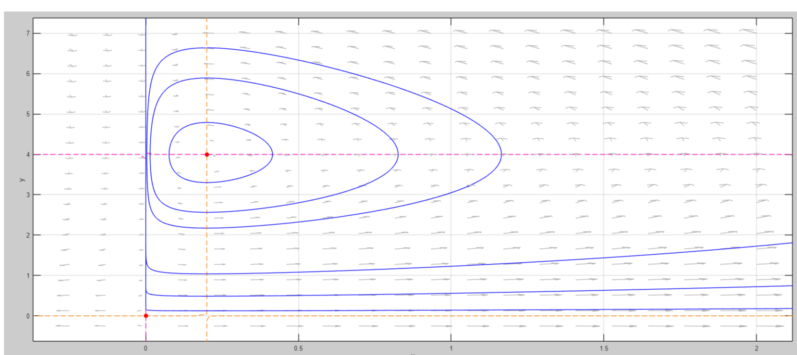

FIGURE 2. Phase Plane of the Phoenix's Population [5]

In Figure 2, the trajectory lines present show that the results for $P_1 and P_2$ are the same as in the dragon's population. We see a saddle node at P_1 and a center for P_2. Since this is the case, the observations are the same for each population. Since Phoenix's residents are fewer in number, there is a possibility that their population will become extinct more quickly compared to dragons. This is without any intervention.

Finally, some distinctive needs to be made regarding how the prey-predator model functions. $\alpha, \beta, \gamma, and \delta$ can be switched between positive and negative values, which changes the equilibrium points' location on the graph and the stability results. This is where Bob will need to do some rearranging of the parameter values to determine if there is a way to increase both populations to help with species conservation.

5. Conclusion

The prey-predator is unrealistic since it is sensitive to perturbations that are caused by the population's periodic behavior [2]. Since there are better models to demonstrate prey-predator behavior, such as models that consider the logistic growth rate. Logistic growth models are constructed to reflect species and their parameters in more detail compared to the prey-predator model.

Some things to take into consideration are the species' gestational development, maturity development, and behavior changes (adaptation). These biological standpoints can change per species and per individual. Taking them into account will increase the difficulty of the models but make them more precise in what the impacts are of removing dragons and phoenixes needed resources. Parameters that will be necessary to see when studying the behavioral changes in animal communities are diseases, refuge, fuzzy, etc.

Using mathematical models to study populations can help provide valuable insights into conserving them properly for future generations. Especially if research cannot be replicated experimentally, it is there to provide a better understanding of what the next best steps are to take concerning conservation.

References

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