When Calculus actually becomes Complex

By: Lorentz Albers

Introduction

While most students interested in Mathematics will come across Complex numbers at some point, or even learn about them in School, I rarely heard them perceived as anything other than annoying calculations and without a real purpose. Despite this, when you read about Analysis you will come across them all the time, often in definitions that University Textbooks widen to include them for what seems to be no apparent reason.

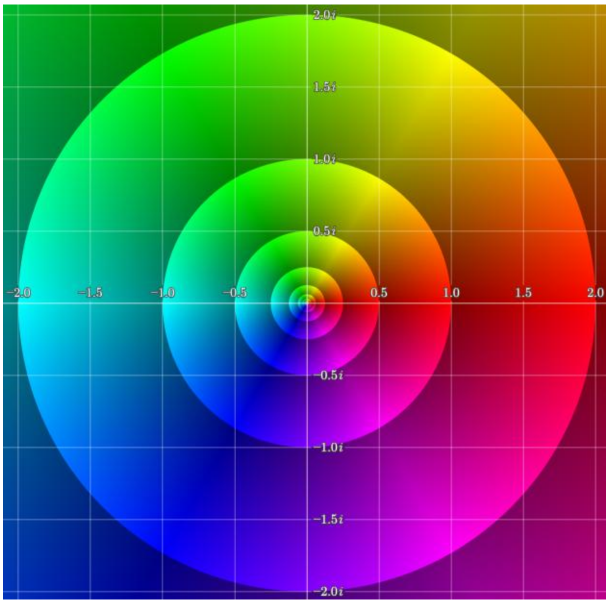
But Complex Analysis can be very useful and especially interesting, which is why I want to discover it in this Essay, building on what most students still learn in School. I also want to integrate a little bit of the notation used in Mathematical Literature and some more abstract topics from other Areas of Mathematics.

Interpreting Complex Functions

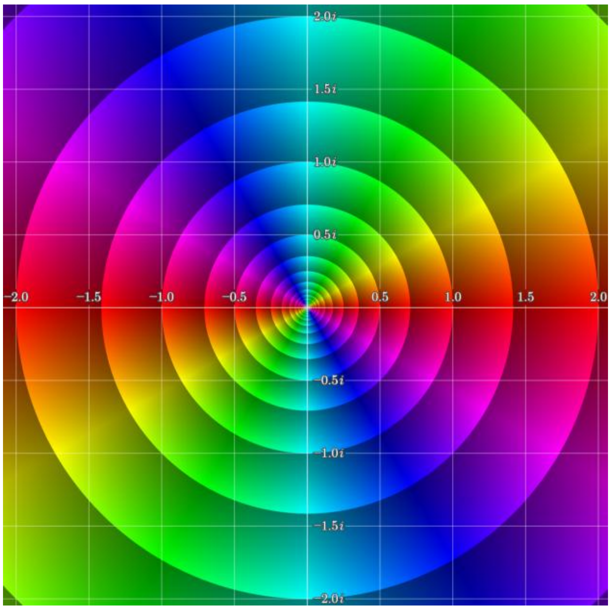
First, we need to look at polar coordinates. They are an alternative representation of complex numbers that use an angle and an absolute value. While they are not suited for calculations, they make it easier to see to what parts a value is real or imaginary. They work based on the unit circle, the construct that brings most students into first contact with complex numbers:

$$z = r \cdot (\cos(\varphi) + i\sin(\varphi)) = r \cdot e^{i \cdot \varphi}$$

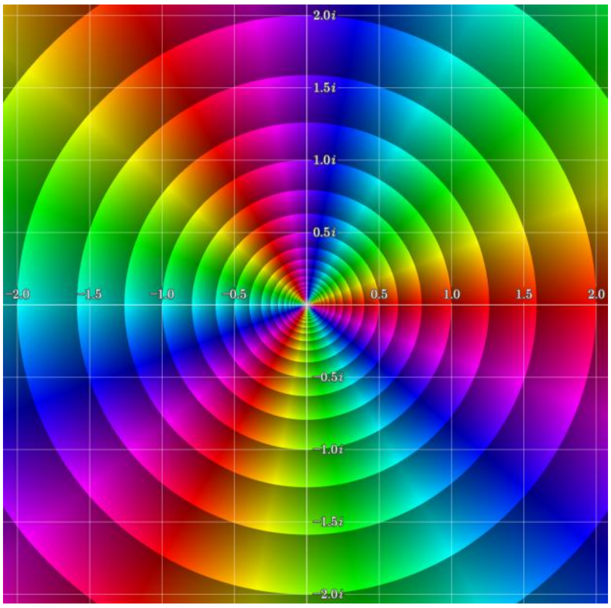
The colour represents the Angle in polar coordinates, and the lines certain absolute values. Red means the values have a positive real part and a comparatively very little or zero imaginary part, light Blue means the same but with a negative positive part. Points where different colours run together are either zeroes or singularities.



To get comfortable with reading complex Functions we will start with interpreting f(z) = z^2.



Of course there is still a zero at the origin. Both sides of the real axis are positive real numbers as it's the same result as with real numbers and along the imaginary axis the numbers are negative, as we expect with $(a \cdot i)^2 = -a^2$. We can also see how there are two roots for any number, each on opposing sites of the origin. By looking at f(z) = z^3 we can so more interesting things.



We have the same pattern, except that every colour can be seen thrice now. This makes the different behaviours of even and uneven exponents seem a lot less arbitrary when you can look at the symmetry in the complex plane. You can also see that there are three numbers for every value, all 120 degrees rotated around the origin from one another. Furthermore, Lines are closer together, showing the faster growth when compared to z^2.

Looking at these standard functions on the complex plane made a lot of things feel a lot more intuitive and lead me to choose this topic for an essay. I can recommend trying around with different exponents and functions in general in a complex function plotter, it's quite fascinating when you haven't dealt with complex functions before.

Holomorphisms and Derivatives of Complex Functions

The derivative of a complex function looks like the derivative of a real function:

Let Ω be an open subset of $\mathbb{C}$, and suppose $\mathrm{f}: \Omega \to \mathbb{C}$ *is a function. We say that f is complex analytic on Ω if f is complex differentiable at each* $p \epsilon \Omega$*. In other words f is complex analytic if at each p the following limit exists:*

$$f'(p) := \lim_{\zeta \to 0} \frac{f(p+\zeta) - f(p)}{\zeta}$$

p.31 (D'Angelo, 2010)

This results in the derivatives generally being the same ones as the real ones, if they exist. Product, quotient and chain rules also hold. Moreover, Complex functions are always infinitely differentiable, meaning you can take the derivative of the derivative and the derivative of that and so on.

Complex functions are called Holomorphic at a point if they are complex differentiable in a neighbourhood of that point (D'Angelo, 2010). While I will not be using that term it is used surprisingly often.

Alternate Notation

Next we will quickly take a look at another Definition of differentiability and the derivative, that before I had mostly seen in the Context of Multivariable Calculus, which focuses more on the linearity of the function when looking at smaller and smaller intervals:

We write

$$f(p+\zeta) = f(p) + f'(p) \cdot \zeta + E(p, \zeta)$$

Where the function E is defined by (4). If the limit [that expresses the derivative] *exists, then E satisfies:*

$$\lim_{\zeta \to 0} \frac{E(p, \zeta)}{\zeta} = 0$$

p.84 (D'Angelo, 2010)

This expression is linear except for an error, which must get infinitely small when the function is differentiable. To get to grasp with this we will look at the example of z^2 again:

$$(z + \zeta)^2 = z^2 + 2z \cdot \zeta + \zeta^2$$

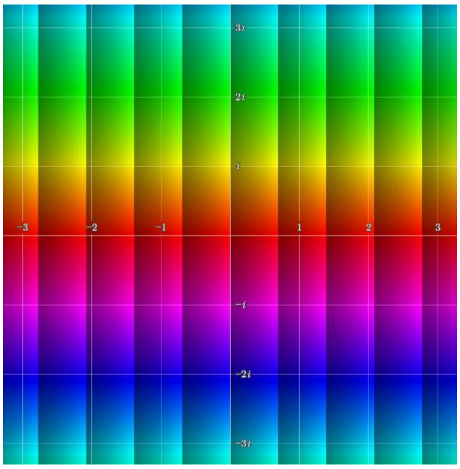
$$\lim_{\zeta \to 0} \frac{\zeta^2}{\zeta} = \lim_{\zeta \to 0} \zeta = 0$$

$$\Rightarrow (z^2)' = 2z$$

While this is interesting, the advantages are not immediately obvious (at least to me initially), however it seems to be used quite a bit in proofs, especially surrounding Analyticity, which I will talk mention again later.

Example

For a simple derivative we will look at the simplest derivative of them all, $\mathrm{f(z)} = \mathrm{e^z} = \mathrm{f'(z)}$



In general lines can be seen along the direction of the real axis. It stays the same the colour at every point between the original and the derivative is the same. This can be explained by paying attention to the fact that moving in the direction of the real axis on the graph results in a growth while keeping the same ratio between complex and real parts, which means that the growth needs to have the same ratio, resulting in the same colour in the diagram. The change of colour along the imaginary axis can also be explained by looking at the unit circle and the behaviour of $\mathrm{e^{ix}}$ which travels in a circle around the origin and in that way defines the colour of the polar coordinates.

There are also many other interesting patterns to spot when looking at complex functions and their derivatives and once again I can recommend playing around with a complex function Plotter.

Comparison to vector valued functions

Similarities between Complex Numbers $\mathbb{C}$ and the Vector space $\mathbb{R}^2$ can be found as well when looking at derivatives. To do this we will use the mapping:

$$g\colon \mathbb{C} \to \mathbb{R}^2, z \to \begin{matrix} Re(z) \\ Im(z) \end{matrix}.$$

It is an isomorphism, meaning that it can be inverted and that it keeps the structure of $\mathbb{C}$ and $\mathbb{R}^2$ intact in terms of addition and (scalar) multiplication, as can be shown with:

$$g((a+b\cdot i)+(c+d\cdot i)) = g((a+c)+(b+d)\cdot i) = \begin{matrix} a+c \\ b+d \end{matrix} = \begin{matrix} a \\ b \end{matrix} + \begin{matrix} c \\ d \end{matrix}$$
$$= g(a+b\cdot i) + g(c+d\cdot i)$$

$$g(x \cdot (a+b\cdot i)) = g(x\cdot a + x\cdot b\cdot i) = \begin{matrix} x\cdot a \\ x\cdot b \end{matrix} = x \cdot \begin{matrix} a \\ b \end{matrix} = x \cdot g(a+b\cdot i)$$

Now we can turn f into a vector valued function $\bar{f}$ by composing $g \circ \mathrm{f} \circ g^{-1}$:

$$\bar{f} := g(f(g^{-1}(\begin{smallmatrix} a \\ b \end{smallmatrix}))) = g(f(a+b\cdot i)) = \begin{pmatrix} Re(f(a+b\cdot i)) \\ Im(f(a+b\cdot i)) \end{pmatrix} \underset{\substack{Re(f(a+b\cdot i))=u(a,b) \\ Im(f(a+b\cdot i))=v(a,b)}}{\Longleftrightarrow} \begin{pmatrix} u(a,b) \\ v(a,b) \end{pmatrix}$$

By substituting $f(a+b\cdot i)$ with $u, v\colon \mathbb{R}^2 \to \mathbb{R}$ we can now use Differentiation of vector valued functions without looking at complex numbers:

$$\bar{f}'(a,b) = \begin{pmatrix} \frac{\partial \mathrm{u}}{\partial \mathrm{a}} & \frac{\partial \mathrm{u}}{\partial \mathrm{b}} \\ \frac{\partial \mathrm{v}}{\partial \mathrm{a}} & \frac{\partial \mathrm{v}}{\partial \mathrm{b}} \end{pmatrix}$$

While our construction of this function doesn't show it this is the same as if we composed $g \circ \mathrm{f}' \circ g^{-1}$ in the same way we did above. For this it is important that deriving the real part just means the real part of the inner derivative.

It works this way around, but now the question becomes, when does it work the other way around, when does the derivative of a function in $\mathbb{R}^2$ equal the complex derivative? For this we can look at the Cauchy Riemann Equations.

$$\frac{\partial \mathrm{u}}{\partial \mathrm{a}} = \frac{\partial \mathrm{v}}{\partial \mathrm{b}}, \qquad \frac{\partial \mathrm{u}}{\partial \mathrm{a}} = -\frac{\partial \mathrm{v}}{\partial \mathrm{b}}$$

We will apply them to z^2 to show how to work with them:

$$\mathrm{u(a,b) = Re((a+ib)^2) = Re(a^2 + 2abi\text{-}b^2) = a^2\text{-}b^2}$$

$$\mathrm{v(a,b) = Im((a+ib)^2) = Im(a^2 + 2abi\text{-}b^2) = 2ab}$$

$$\frac{\partial}{\partial \mathrm{a}}(\mathrm{a^2\text{-}b^2}) = 2\mathrm{a} = \frac{\partial \mathrm{v}}{\partial \mathrm{b}}(2\mathrm{ab}), \qquad \frac{\partial}{\partial \mathrm{b}}(\mathrm{a^2\text{-}b^2}) = -2\mathrm{b} = -\frac{\partial}{\partial \mathrm{a}}(2\mathrm{ab})$$

As we can see z^2 fulfils the Cauchy Riemann Equations, which is expected as this along with being real differentiable is equivalent to the function being complex differentiable. This way we can use Real Analysis to proof complex Differentiability.

Epilogue: Analytical Functions

Another property of complex differentiable functions is that they are so called Analytical Functions. This property also exists in real analysis but plays a much more important role in complex analysis. Even though the formal definition is more general, it can be looked at based on the Taylor series that one might come across in School:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

If it exists and converges to the value of the function at a point, meaning that the function can be approximated by a polynomial, the function is analytical at that point. This property allows functions to be expanded beyond their initial Domain in a consistent way. For complex functions they are a lot more relevant because when fulfilling few conditions there already is a clear way to continue the function. Applying this is important for many problems in Analysis, most famously the Riemann-Hypothesis, which I will not go into but can recommend you look into.

Conclusion

Complex Analysis is an area that, unsurprisingly, is quite close to Real Analysis. Here we were able to go from Real Analysis to Complex Analysis quite easily without doing much differently and were able to learn about a topic that becomes quite important later on when studying Mathematics. While I think that this is not a topic that is particularly original, I hope I was able to convey the fascination it gave me when I first started to get to grips with it.

Literature

D'Angelo, J. P. (2010). *An Introduction to Complex Analysis and Geometry.* American Mathematical Society.

Internet Sources

https://samuelj.li/complex-function-plotter/ (Images)