

The Genius of Euler's Formula

Euler's formula is one of the most famous formulas in mathematics. It provides a clean and simple, yet counterintuitive and difficult to discover, connection between exponentiation and trigonometry. In specific cases, it also forms connections between the rationals and irrationals, the reals and the complex. It has incredible applications in other fields of mathematics as well, often forming beautiful proofs that range between real and complex numbers throughout. These reasons and many more help to show just how genius Euler's formula really is.

If someone says that Euler's formula is obvious, they are either lying or are at such a mathematical level that they and everyone around them take it for granted. Now, in higher level mathematics obvious and trivial often mean something closer to "I don't want to show or explain how or why I did this because someone else has done it before", but before the collegiate level the students will expect a much more typical definition for the word obvious. And clearly such a definition does not hold for something so unintuitive as Euler's formula, as it is a connection between a few forms of mathematics that high schoolers often struggle with, exponentials with non integer arguments, trigonometry, and complex analysis. Throughout all of high school and lower mathematics, we are taught that exponential functions only increase and we never see any form of periodicity in them, while we see periodicity as a defining feature of the trigonometric functions. We also keep away from complex numbers for almost the entirety of high school, maybe briefly discussing the definition of the imaginary unit, i , in an Algebra II course. Many high school students completely shut down when i is included in a problem, especially one as difficult to complete without guidance for the first time as providing a derivation for Euler's

formula. Even second semester calculus students, who know the tools to derive the formula, almost never find this relationship without prompting at multiple steps along the way. The formula is also unintuitive because of the connection between such broad fields of mathematics that it forms. By combining such broad fields and doing things we are not used to seeing with the involved function families, Euler's formula becomes not only a tricky thing to understand, but also a tricky thing to use. However, these attributes that make Euler's formula so tricky are also the attributes that make it so useful for mathematicians the world around.

Euler's formula connects three fields of math that seem unrelated at first glance, exponentiation, complex analysis, and trigonometry. Though it is reaching a bit far to refer to exponentiation as its own field, it is a topic that can be independently studied to look at more complicated exponential. By taking Taylor series and identifying sub-series, it is clear that Euler's formula holds and that it forms a strong bridge between complex numbers and trigonometry, which allows mathematicians to take complex exponentials and to evaluate trigonometric functions with complex arguments. By rearranging the formula, one arrives at a complex exponential function for sine or cosine that reminds us of the exponential function for hyperbolic sine or cosine. These formulas can be easily verified by substituting Euler's formula back into the equation and also allow for complex analogs to the secant, cosecant, tangent, and cotangent functions. These complex analogs allow for answers to many questions that pre calculus students have often tried to stump their classmates with, such as $\sin(x)=2$. Along with the clear trigonometric relations given by the formula, there are also implications of the formula for logarithmic functions. Mathematicians take logarithms of complex numbers by expressing a complex number in the form of $re^{i\theta}$ and then taking the logarithm of that as $\ln(r)+i\theta$. This

allows for logarithms to be expanded into all numbers other than 0. This incredible connection between such broadly different branches of mathematics is a clear sign of the genius behind Euler's formula.

For such an interesting formula, anyone would expect it to have some very interesting uses, and Euler's formula easily delivers on that expectation. For an easy example, it is important to one method of solving the Basel problem, one of Euler's most famous accomplishments. This is a quote about the solution given by Euler himself:

“What a marvelous derivation it was. It boasted an all-star cast of transcendental functions: sines, cosines, logs, and exponentials. It ranged from the real to the complex and back again. It featured L'Hôpital's Rule in a starring role. Of course, none of this would have happened without the fluid imagination of Leonhard Euler, symbol manipulator extraordinaire.”

—William Dunham

This clearly shows the genius application that Euler's formula is capable of, but it is also capable of further application and applications in other places. Euler's formula can be used to find expansions of the trigonometric functions, such as $\sin x = \frac{e^{ix} - e^{-ix}}{2i}$. These formula can then be used to find trigonometric derivatives that, while on the surface look more complicated, provide a simple understanding of why sine and cosine have the property where their derivatives operate in a cyclical fashion, which many calculus students do not understand very well. Euler's formula

also holds many applications in the field of complex analysis. Some standout applications include the conversion of complex numbers in Cartesian form to a polar form of $re^{i\theta}$, which allows for easy multiplication and logarithms with complex bases and arguments. One of the most important uses of Euler's formula in complex analysis is its use in the Fourier transform, which is used for signal processing, data compression, and more.

It has often been said that $e^{i\pi} + 1 = 0$, a specific case of Euler's formula for $\theta=\pi$, is the most beautiful equation in all of mathematics and it would not be possible without Euler's formula. The most common reason people cite for the beauty of this equation is that it connects the five most important constants in mathematics, 0, 1, e, π , and i. By connecting these constants Euler's identity allows for a quick conversion from complex numbers to reals and to add and subtract complex exponentials. But I posit that there is a more important reason for the beauty of this equation, the incredibly simple connection between reals and complex numbers, rationals and irrationals. It is very rare to see such a simple connection between numbers at all, and it is very hard to make simple connections between real and complex numbers other than the simple basis of complex numbers, $i^2 = -1$. It is exceedingly beautiful for an equation to not only make this connection, but also to connect the rational and irrational numbers, something rarely seen other than simple roots and the accompanying exponents. By making these incredible connections between such broad forms of numbers and at the same time connecting such important constants, Euler's identity cements itself as the most beautiful equation in the entirety of mathematics.

The incredibly unintuitive nature of Euler's formula, and the connections it makes between multiple fields of mathematics help show the incredible genius behind Euler's formula. It is extremely difficult to form connections between these fields and even more difficult to do so cleanly, so the simple connection formed by Euler's formula in turn shows further genius in its ability to do so. With such broad implications from such a simple statement, and the broad connection between the different fields of mathematics, it is clear that Euler's formula is the most genius formula in all of mathematics.