Expanding the Eight Queens Problem

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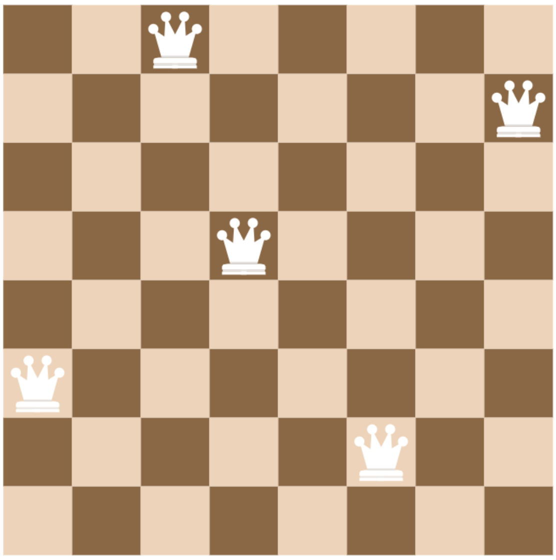

Figure 1: Five queens which 'dominate' the board.

1 Introduction

Chess is perhaps the most well-known game of all time. It has a ton of depth and strategy, and players are still learning more about the game centuries after the modern variant was introduced. But today, I'm not going to talk about any of this. Instead, we will be preventing any battles as efficiently as possible. On that note, can you place eight queens on a chessboard such that no two queens attack each other?

Problem 1.
Well, can you?

It turns out that you can. Here is one solution:

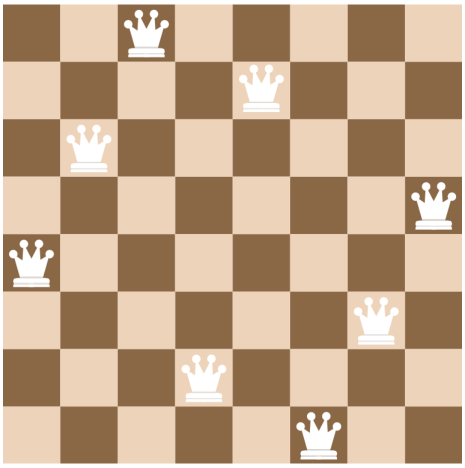

Figure 2: One of the solutions

A normal puzzle enjoyer might leave it at that. But I am not an ordinary puzzle solver. I am a mathematician. Which means that after solving this problem, instead of being satisfied, I came up with some more questions to expand on this problem:

A) How many solutions are there to the eight queens puzzle?

B) What about chessboards of a different size? How many queens can be put on those?

C) What about other chess pieces?

In this essay, I will try to answer all of these problems. We will start the third question.

2 Rookie puzzles

The first chess piece we will look at is the rook.

2.1 Rooks

Problem 2.
How many rooks can be placed on a chessboard, such that no two rooks attack each other?

It is pretty easy to tell that the answer must be 8. There can at most be one rook on each row, and a chessboard has only 8 rows, so we know that we cannot place more than 8 rooks. But we can place exactly 8 rooks, for example, by placing them alongside a long diagonal:

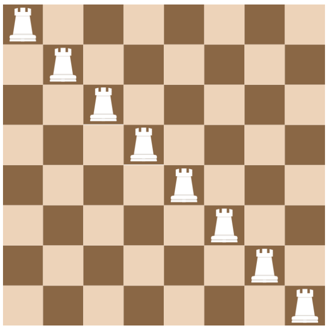

Figure 3: Rooks on a diagonal

We can also answer question B for rooks in a similar way. Suppose we have an $n \times n$-chessboard. Such a board has only n rows, and every row can at most have one rook. So we can at most place n rooks. Placing n rooks is possible: for example, by placing them on a long diagonal.

2.2 Kings

The second-easiest piece to solve is kings. Let's take a look at it.

Problem 3.
How many kings can be placed on a chessboard, such that no two kings attack each other?

Your intuition will probably tell you that the answer must be 16. Finding a solution for 16 kings isn't very difficult. Here's one:

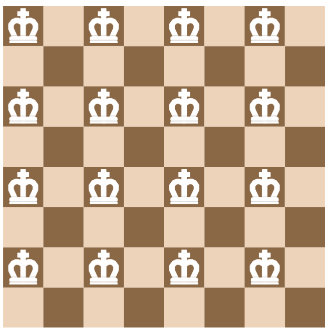

Figure 4: 16 kings on a chessboard

Proving that this is the maximum is a little trickier, but still doable. We can use the fact that if a king is in a 2×2-square, it attacks all three of the other squares:

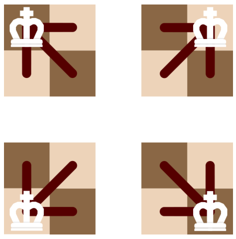

Figure 5: Attacks of kings in 2×2-squares

Therefore, every 2×2-square has at most one king. Since we can divide a chessboard into 16 2×2-squares, we can at most place 16 kings on a chessboard.

With similar logic and solutions, we find that we can place at most k^2 kings on $(2k - 1) \times (2k - 1)$ and $2k \times 2k$ chessboards.

2.3 Bishops

Problem 4.

How many bishops can be placed on an $n \times n$-chessboard ($n \in \mathbb{N}$), such that no two bishops attack each other?
(Two bishops attack each other if they are on the same diagonal.)

This time, we immediately start with the generalized version. However, it is still a good idea to look at small chessboards, to get a better understanding of the problem. Let's look at a standard 8×8-chessboard.
After trying this for a while, you might find something like this:

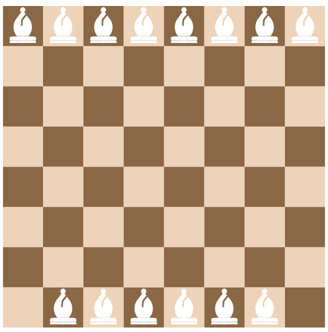

Figure 6: 14 bishops on a chessboard

It turns out that 14 is the maximum. We can prove this by counting the number of diagonals that go from top-left to bottom right. There are 15 of these $(2 \cdot 8 - 1)$, one for each square on the left and top side of the board:

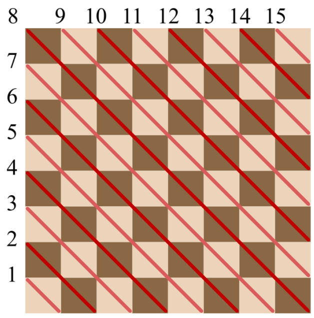

Figure 7: The 15 top-left to bottom-right diagonals of a chessboard

Each one of these diagonals can have at most one bishop, so that means we cannot place more than 15 bishops on a chessboard. However, we still have to prove that 15 is also not possible. So how can we do that?

Well, if we were to place 15 bishops, each diagonal must have one bishop. However, we can see that two of the diagonals only cover one square. So both of those squares must have a bishop. However, it turns out these squares are the top-right and bottom-left corners, which are on the same diagonal that goes from top-right to bottom-left. So that means we just placed two bishops on the same diagonal! That means it is impossible to place 15 bishops, therefore 14 bishops is the maximum.

Using the same ideas, we find that on an $n \times n$-chessboard, we can place at most $2n - 2$ bishops ($2n - 1$ diagonals of which at least one must be empty because of the corner argument. Also, if $n = 1$, the maximum is 1).

Now it is time to move on to trickier pieces.

3 Diagonal dominoes

The next chess piece we will cover is the knight.

3.1 Knights

Problem 5.
How many knights can be placed on an $n \times n$-chessboard ($n \in \mathbb{N}$), such that no two knights attack each other?

Some chess players might know that a knight must always move to a square of a different color. That means that if we place all knights on squares of the same color, they will not attack each other. This means that we can place $\lceil \frac{n^2}{2} \rceil$ knights on an $n \times n$-chessboard (and if $n \leq 2$, each square can have a knight).

Proving that this is the maximum is quite tricky. The last few times, we proved a maximum by using shapes (rows, diagonals, 2×2-squares) which could only have one of a certain chess piece. For knights, we need to have a shape of two squares which can divide up the board and can at most have one knight.

For this, I present: the knight domino!

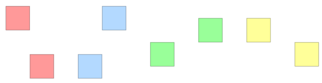

Figure 8: The knight domino

Now we need to divide the entire board up into these (for odd values of n, we can leave the middle square out). These are the solutions for $3 \leq n \leq 6$:

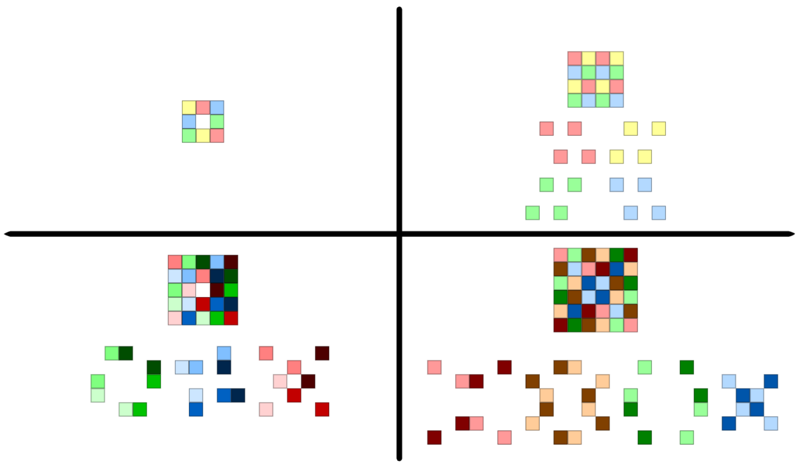

Figure 9: Solutions for $n = 3$, $n = 4$, $n = 5$ and $n = 6$.

We can also make the following shapes:

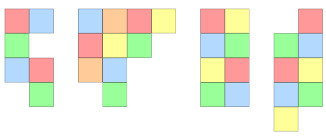

Figure 10: Some pieces made out of knight dominoes.

With these shapes, we can make a ring of two squares wide around any square with sides of length 3 or greater. This means that if we can cover an $n \times n$-chessboard with knight dominoes (except for the very center square), we can cover an $(n+4) \times (n+4)$-chessboard as well. Since we can cover $n \times n$-chessboards if $3 \leq n \leq 6$, we can cover all chessboards with side-lengths of 3 or greater with knight-dominoes, proving that the maximum is indeed $\lceil \frac{n^2}{2} \rceil$.

3.2 Pawns

In chess, pawns have some special rules. So before we try to find the maximum, we should first look at which rules we will use for pawns. The rules I used are that pawns cannot go on the top or bottom row and that if their squares share exactly one corner, then one of the pawns is attacking the other. With that out of the way, try to solve the next problem yourself! Take advantage of the strategies I used.

Problem 6.
How many pawns can be placed on an $n \times n$-chessboard ($n \in \mathbb{N}$), such that no two pawns attack each other?

However, with this, we are not done. We still have to look at queens, but first...

4 Wait, there's more?

We have not yet taken a look at question A, so let's do that now.

4.1 Rooks, revisited

Problem 7.
In how many ways can the maximum number of rooks be placed on an $n \times n$-chessboard ($n \in \mathbb{N}$), such that no two rooks attack each other?

(Fun fact: this is a table and therefore doesn't contribute to the word count!)

We need to place n rooks, one on each row.

In the first row, we can place a rook in n places. For the second row, we have $n-1$ options, as the previous rook blocks one column. On the third row, we have $n-2$ options, then $n-3$ and so on, until row n where we have one option left.

This gives $n \cdot (n-1) \cdot (n-2) \cdots 1 = n!$ solutions. (Yes, factorials make a cameo in chess puzzles)

4.2 The proof is left as an exercise to the reader

What about the other chess pieces covered in chapters 2 and 3? Well, they provide some very interesting and cool puzzles. I could talk about how powers of two are related to bishops, or how the choose function shows up in the even n case for pawns, or how I haven't been able to solve the even n case for kings yet! But the word limit forces me to leave these problems as exercises for you, the reader. Try them! They are ordered by difficulty.

Problem 8.
In how many ways can the maximum number of kings be placed on a $(2n-1) \times (2n-1)$-chessboard ($n \in \mathbb{N}$), such that no two kings attack each other?

Problem 9.
In how many ways can the maximum number of pawns be placed on a $(2n-1) \times (2n-1)$-chessboard ($n \in \mathbb{N}$), such that no two pawns attack each other?

Problem 10.
In how many ways can the maximum number of knights be placed on an $n \times n$-chessboard ($n \in \mathbb{N}$), such that no two knights attack each other?

Problem 11.
In how many ways can the maximum number of bishops be placed on an $n \times n$-chessboard ($n \in \mathbb{N}$), such that no two bishops attack each other?

Problem 12.
In how many ways can the maximum number of pawns be placed on a $2n \times 2n$-chessboard ($n \in \mathbb{N}$), such that no two pawns attack each other?

Problem 13.
In how many ways can the maximum number of kings be placed on a $2n \times 2n$-chessboard ($n \in \mathbb{N}$), such that no two kings attack each other? (I don't know if this is solvable)

5 The return of the queen

Now it's time to take another look at queens on chessboards. First of all, let's answer question A: how many ways are there to place eight queens on a chessboard?

5.1 All solutions to the eight queens puzzle

In the last chapter, we figured out that there are $8! = 40320$ ways to place 8 rooks. Since queens are just rooks with diagonal attacks added, a solution for queens must also be a solution for rooks. That means all solutions are in the 40320 solutions for rooks. If we had a computer, we could simply check all these options to find all solutions.

We can also solve this problem by hand, but in order to do that, we need to be more efficient. This is the structure of the proof that I came up with:

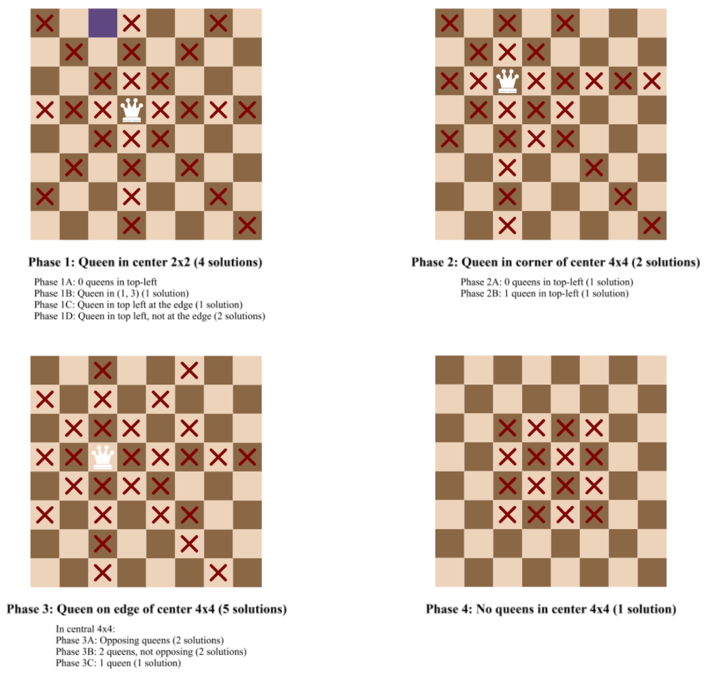

Figure 11: Structure of the proof

With this we find 12 fundamental solutions, which give 92 solutions in total (using rotations and flips).

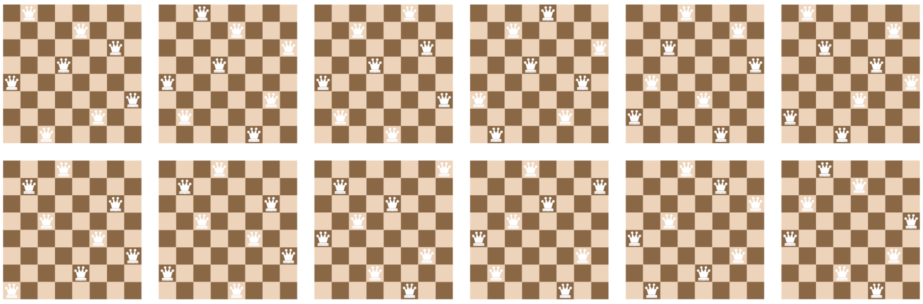

Figure 12: The twelve fundamental solutions

5.2 The n-queens puzzle

With that out of the way, it is time to answer question B once and for all:

Problem 14.
How many queens can be placed on an $n \times n$-chessboard ($n \in \mathbb{N}$), such that no two queens attack each other?

First of all, using the same argument as for rooks, we find that we can at most place n queens on an $n \times n$-chessboard. The challenging part this time is to find a construction for n queens.
Once again, it is a good idea to start with smaller chessboards. We find that only one queen can be placed if $n = 1$ or $n = 2$, and that only two queens can be placed if $n = 3$.

For $n = 4$, we have this solution:

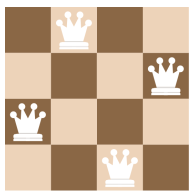

Figure 13: Solution for $n = 4$

This solution doesn't immediately provide ideas for a general construction, so let's keep looking.
For $n = 5$, most solutions look something like this:

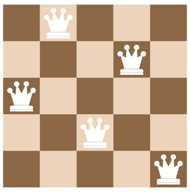

Figure 14: Solution for $n = 5$

It's the solution of $n = 4$, but with a queen in the new corner! Would this work for other values of n?
It's obvious that placing a queen on the newly added corner will be on a unique row and column compared to the other queens. One of its diagonals is only the square itself, and the other is a long diagonal of the original square. So if we have a solution for n which leaves one long diagonal open, $n + 1$ is possible as well. So now we just have to think about even n, where we need to find solutions with at least one long diagonal unfilled. Let's keep looking.
Here's a solution for $n = 6$:

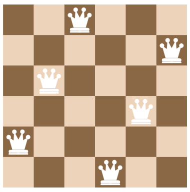

Figure 15: Solution for $n = 6$

It looks like two staircases of knight moves! Would something like this work for other n?

It's obvious that this solution has all queens on different rows and columns. One set of diagonals also works for any even n:

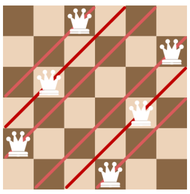

Figure 16: The set of diagonals that cannot interfere

The other set also works most of the time. It creates two distinct sets of diagonals, one for each staircase, where each third diagonal is in the set:

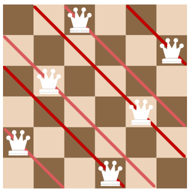

Figure 17: The set of diagonals that can interfere

For most even n, these two sets will not overlap. But for some, they do:

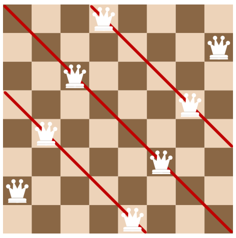

Figure 18: Two sets of diagonals that interfere

It turns out that if (and only if) $n \equiv 2 \mod 3$, the two sets of diagonals interfere. All n that are equivalent to $0, 1, 4$ or $5 \mod 6$ therefore work. But for $n \equiv 2 \mod 3$, we must look further.
However, 8 is an even number, and $8 \equiv 2 \mod 3$. And we already know all solutions for $n = 8$. All we have to do is generalize one of these solutions.
Doing this is not easy however. In fact, when I first tried this last summer, I couldn't do it. But after looking after it again six months later, I managed to solve it.

The breakthrough was that I realized that this solution:

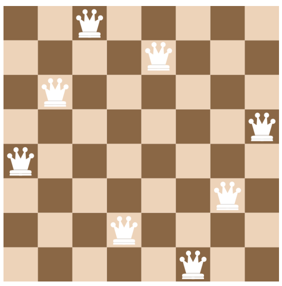

Figure 19: One of the solutions

looks like it has three staircases: two are normal, but the middle one is a little stretched.

The question is, can we extend these staircases to get solutions for $n = 14$, $n = 20$ and so on? Well...

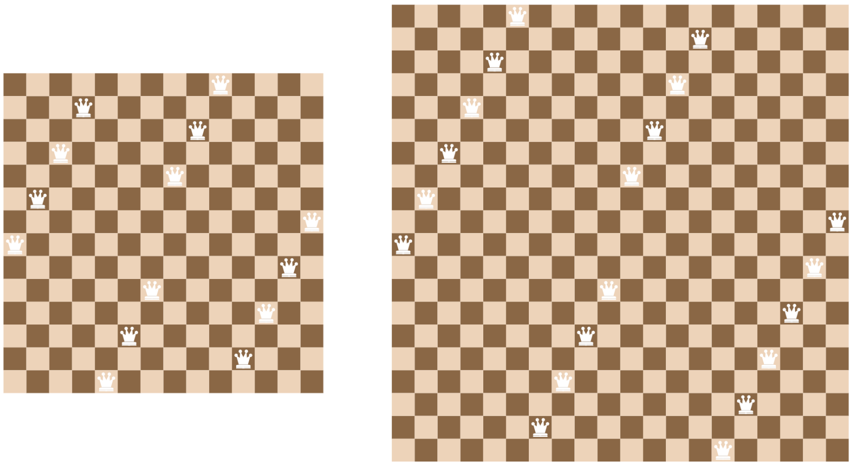

Figure 20: The solutions for $n = 14$ and $n = 20$

For all other even $n \equiv 2 \mod 3$, we can keep extending the staircases in the same way. And with that, we have proven that we can place n queens on an $n \times n$-chessboard (except if $2 \leq n \leq 3$).

Now there is one more question, how many solutions are there to placing n queens on an $n \times n$-chessboard? And the answer is... we don't know! For $n \geq 28$, the amount of solutions is still unknown. That means we have managed to get to an unsolved problem in mathematics, simply by expanding a chess puzzle.

6 Conclusion

So what does all of this mean?

Mathematicians were already interested in the eight queens puzzle right after its creation. For example, Gauss has worked on this puzzle, and the solution for the n-queens problem was presented in a mathematical paper in 1874. This shows that puzzle solving and mathematics are closely related. You could maybe even say that mathematics *is* puzzle solving. And that is what makes it so fun!

So what about factorials, powers of two and the choose function showing up? Well, that shows that mathematics is very interconnected, sometimes in very surprising ways, making it all the more interesting.

So in conclusion, I think this essay shows that mathematics is beautiful.

7 Answers

6. $(n-2) \cdot \lceil \frac{n}{2} \rceil$ pawns
8. 1
9. 1
10. 2 if $2 \mid n$, 1 if $2 \nmid n$, there are some exceptions for $n \leq 4$.
11. 2^n
12. $\binom{2k-1}{k}^2$
13. -

8 References

1. Wikipedia. "Eight queens puzzle." Accessed: 13 April 2026.
URL: https://en.wikipedia.org/wiki/Eight_queens_puzzle