An Illusion of Smoothness: Proof of Skin's Untold Complexity

Utso Chaudhuri
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INTRODUCTION

Tring-Tring, your alarm rings, you wake up and push yourself to your sink and BAM! There it is staring you in your face, that damn pimple is glaring you back in the mirror. You go in closer to get a good look for yourself and what you see now is even worse. Many others and I face this problem and we wish our skin could look as smooth as this:

[1]

But zoom in on even the clearest skin you'll have creases and textures pop up; guess you could say it's a real pore-spective! For reader discretion I am not going to put up an image (you can look it up yourself, it's slightly unsettling). Then why are some people blessed with such good skin when zoomed in, all our skins look similarly horrifying? It comes down to how "smooth" it looks from far away. But what does it even mean for a surface to be "smooth"?

To simplify our problem, we can think of the skin surface as a 2d line and then try to model this line using functions. So now, what makes a function smooth? Well, I can think of 2 mathematical properties, Continuity and Differentiability. Let's start with continuity on all points:

$$f \text{ is continuous at all points if } \forall c \epsilon \mathbb{R} \lim_{x \to c} f(x) = f(c)$$

In simple terms, the function has no jumps or breaks in it (just like skin). Now for differentiability:

$$f \text{ is differentiable at all points if } \forall c \epsilon \mathbb{R} \lim_{h \to 0} \frac{f(c+h) - f(c)}{h} \text{ exists}$$

In simple terms, every point on the function needs to have a unique and real tangent line or if zoomed in enough every point locally should be straight. (unlike skin, as skin is never smooth)

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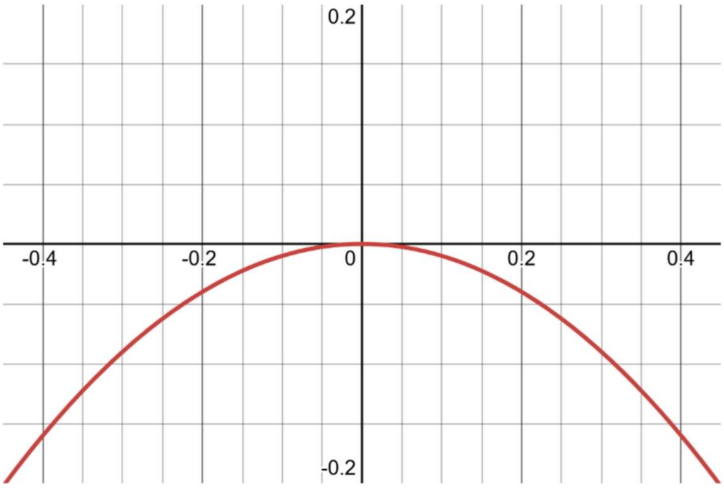
[1] Racho, Janeca. "How to Get Clear Skin: 13 Dermatologist-Approved Tips." *Dermstore*, 21 Oct. 2024, www.dermstore.com/blog/how-to/how-to-get-clear-skin.

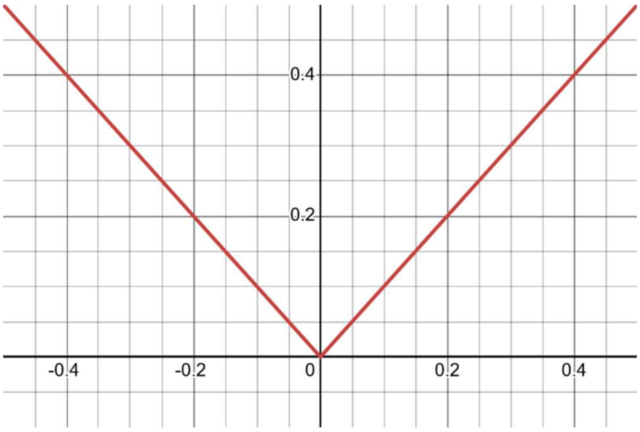
Ok, let's start cooking with some functions now. Take $y = -x^2$:



It's continuous at every point and you can draw a tangent at every point so it's differentiable too. Just like how our skin looks from a distance, $-x^2$ does too. But is it the reality? When zooming into even the clearest of skins, smoothness falls apart. How do we question this mathematically? We can try that by figuring out whether the statement:

$$\exists\ some\ function\ f\ s.t. \forall c \epsilon \mathbb{R}\ f\ is\ continous\ at\ c\ and\ \exists d \epsilon \mathbb{R}\ for\ which\ f\ is\ not\ differentiable$$

is true or false. In simple terms we are asking whether there exists some function that's continuous on all points but non-differentiable on some point. Well, can you come up with such a function (hint: think of zigzags)? That's right! The simplest function would be $f(x) = |x|$ which looks like this:



where $|x|$ is defined as:

$$f(x) = |x| = \begin{cases} -x, & x < 0 \\ x & x \geq 0 \end{cases}$$

So, clearly from the graph itself and the definition for $|x|$ we can tell that $f(x)$ is continuous. But what about non-differentiability? Not obvious? That's the kink! We can use the definition of differentiability to prove this, as at $x = 0$:

$$f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{f(h) - f(0)}{h}$$

Well for $h \geq 0$:

$$f'(0) = \frac{h - 0}{h} = 1$$

And for $h < 0$:

$$f'(0) = \frac{-h - 0}{h} = -1$$

As tangent at point $x = 0$ is non-unique, $|x|$ at $x = 0$ is non-differentiable.

But skin doesn't look like $|x|$, it's more complex, more wavy, more funky. How about we make our statement stricter then:

$$\exists\ some\ function\ f\ s.t. \forall c \epsilon \mathbb{R}\ f\ is\ continous\ at\ c\ and\ f\ is\ not\ differentiable\ at\ c$$

In simple terms, there exists a function that's continuous everywhere but differentiable nowhere. Intuitively, you would think the answer to this statement must be no right? Well, so did mathematicians for a long time, until Karl Weisertrass introduced his coveted Weierstrass functions that did exactly that in 1872.

Simply, the Weierstrass function is defined as:

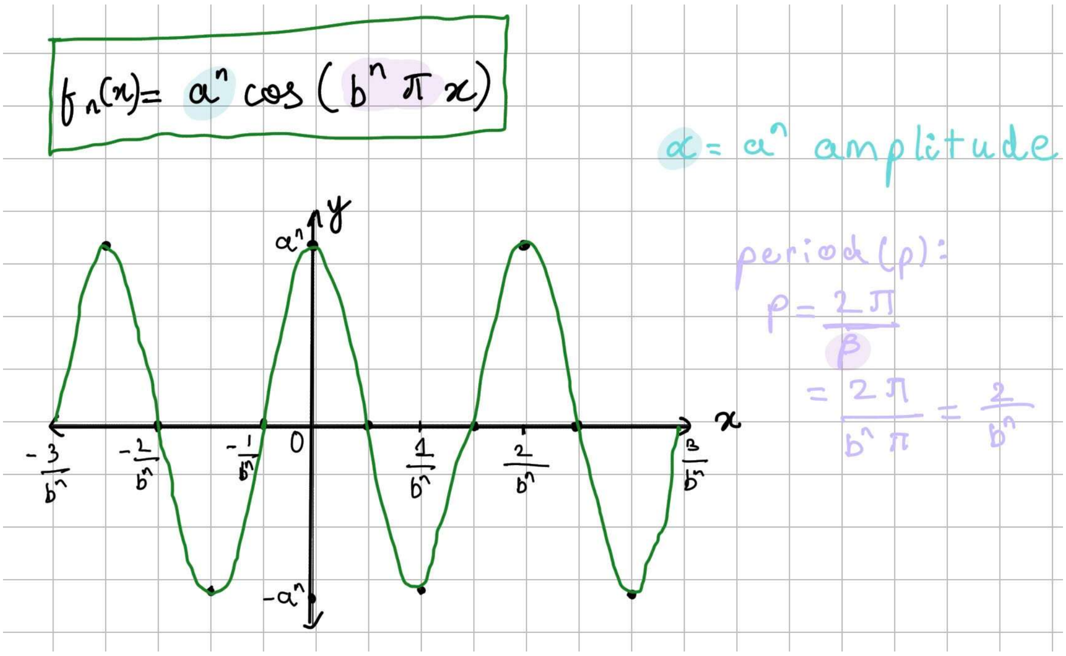
$$W(x) = \sum_{n=0}^{\infty} a^n \cos(b^n \pi x) = \sum_{n=0}^{\infty} f_n(x)$$

Where $0 < a < 1$, b is a positive odd integer, $ab > 1 + \frac{3}{2}\pi$ and $f_n(x) = a^n \cos(b^n \pi x)$.

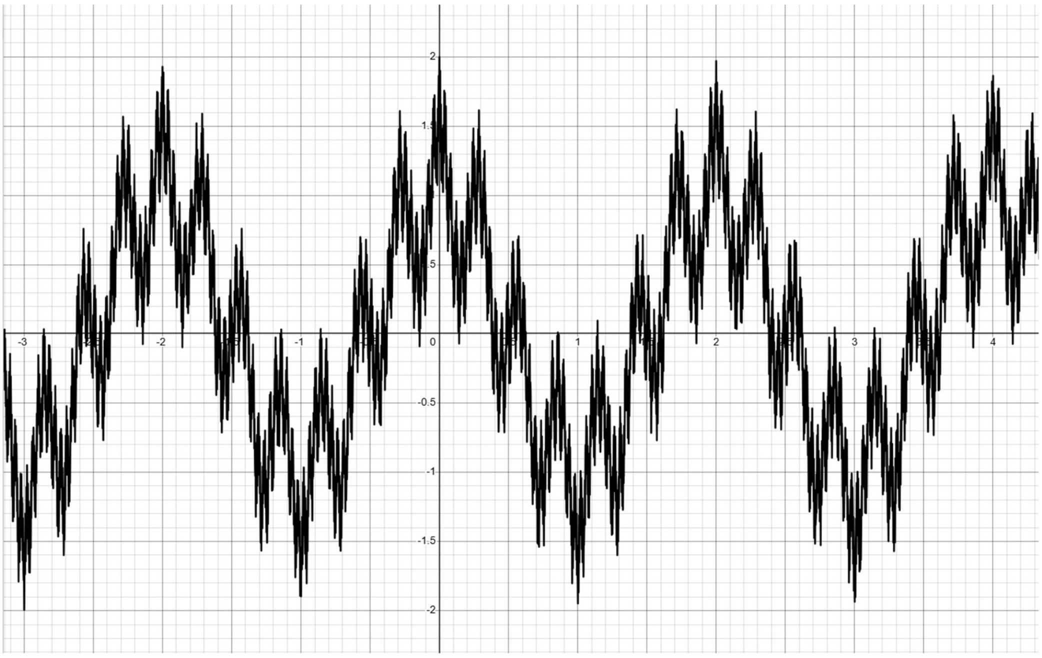
While it looks complex, bear with me, it will be made simple. Ignore the summation for now, let's focus on:

$$f_n(x) = a^n \cos(b^n \pi x)$$

If you noticed, this is of the format $\alpha \cdot cos(\beta x)$ which is just the cos curve's amplitude increased by $\alpha = a^n$ and period changed by $\frac{2\pi}{\beta}$ so a complete oscillation appears every $\frac{2}{b^n}$ for each cos curve, which looks like this:

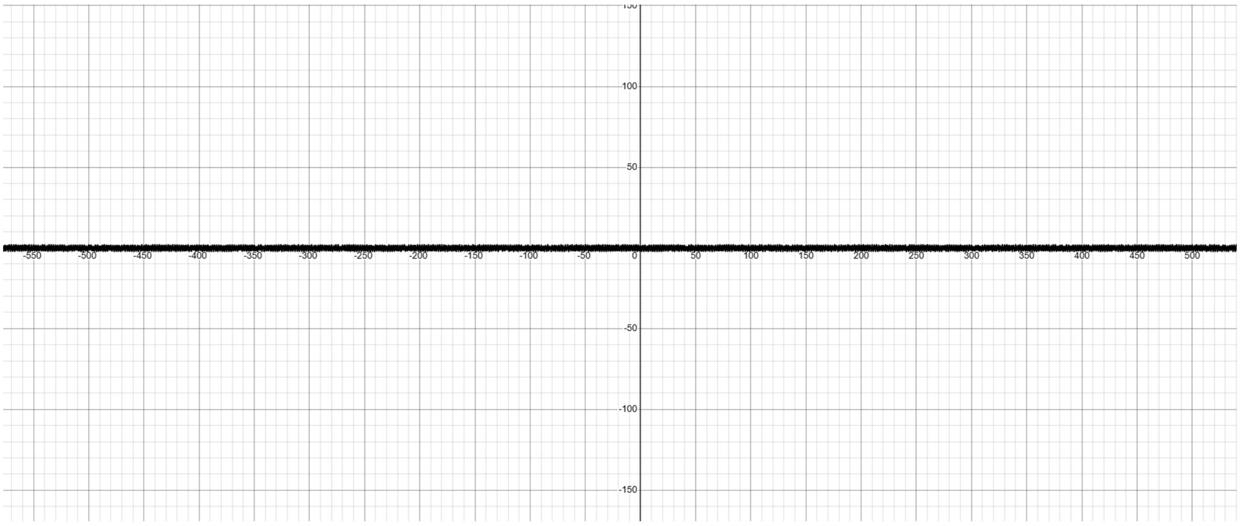


The complexity creeps in with the summation to infinity, with each cos curve added on top of each other. You do it enough number of times and the graph looks like this:



So spiky and never-ending! If you zoom in further, you can see the “wiggles” never stop. Don’t believe me? Go crazy yourself by zooming in (https://www.desmos.com/calculator/bmafitbg0q), you’re never going to stop as the wiggles also don’t!

But if you look from far away, you’d see the true smoothness, the real beauty:



Looks smooth right? So, by the looks of the curve we can guess that yes maybe this function is continuous and non-differentiable at all points (we can guess non-differentiability as even infinite zooming in will not give a clear and unique real tangent at any point).

But we’re mathematicians, we don't guess we prove things. Let’s start by proving continuity at all points. Bear with me this ride’s going to get a bit bumpy!

CONTINUITY PROOF

To start with, the infinite sum in $W(x)$ seems complicated so, let’s simplify with a finite sum:

$$S_N(x) = \sum_{n=0}^{N} f_n(x)$$

Where N is some very large number.

While $W(x) \approx S_N(x)$ can be said, $W(x) \neq S_N(x)$ will always be true, with the absolute error (E) between the two being:

$$E = |W(x) - S_N(x)| \tag{1}$$

$$E = \left|\left(\sum_{n=0}^{\infty} f_n(x)\right) - \left(\sum_{n=0}^{N} f_n(x)\right)\right|$$

$$E = \left|\sum_{n=N+1}^{\infty} f_n(x)\right|$$

For future simplification, let’s consider the value of $|f_n(x)|$:

$$|f_n(x)| = |a^n \cos(b^n \pi x)| \ \leq \ a^n$$

This is true as:

$$\forall \theta \in \mathbb{R}, 0 \leq |\cos(\theta)| \leq 1$$

So, using (1) we can write:

$$E = \left|\sum_{n=N+1}^{\infty} f_n(x)\right| \leq \sum_{n=N+1}^{\infty} |f_n(x)| \leq \sum_{n=N+1}^{\infty} a^n$$

The first inequality follows from the triangle inequality: the size of a sum is at most the sum of the sizes of its terms. (Read it slowly, it will hit you!)

Let’s try and breakdown $\sum_{n=N+1}^{\infty} a^n$ now. If you are sharp, you will notice this is nothing but a geometric series where our first term is a^{N+1} and the terms are increasing by ratio a. Using the sum to infinity formula we get:

$$\sum_{n=N+1}^{\infty} a^n = \frac{a^{N+1}}{1-a}$$

Hence, we can say our error is:

$$E \leq \frac{a^{N+1}}{1-a} \tag{2}$$

One crucial detail about $\frac{a^{N+1}}{1-a}$ is that it is independent of x and only dependent on N. We will make use of this later.

Now consider some epsilon $\varepsilon > 0$, such that for all ε we can choose some N for which:

$$\frac{a^{N+1}}{1-a} < \epsilon \ is \ true \tag{3}$$

How? Well:

$$choose \ N \ to \ be \ arbitrarily \ large, so \ as \ N \to \infty, \frac{a^{N+1}}{1-a} \to 0$$

As a lies between 0 and 1. Therefore whatever the value of epsilon is we can always choose a very large value for N to make $\frac{a^{N+1}}{1-a}$ smaller than ε.

Why did we do this? For comfort. So, using (1), (2) and (3):

$$E = |W(x) - S_N(x)| \leq \frac{a^{N+1}}{1-a} < \varepsilon$$

Upon flipping signs inside the modulus:

$$E = |S_N(x) - W(x)| \leq \frac{a^{N+1}}{1-a} < \varepsilon \tag{4}$$

All in all, we can conclude that:

$$\forall \varepsilon > 0, \exists N \ s.t. |S_N(x) - W(x)| < \varepsilon, \forall n \geq N \ \forall x$$

Sorry for throwing this string of symbols out like a grenade, but it just means for all epsilon greater than zero, there exists some N such that absolute error $E < \varepsilon$ is true for all $n \geq N$ and for all x.

Well, using (4) we have already proved the first half and if you remember in our finite summation of $S_N(x)$ we chose n as $n = N + 1$ so $\forall n \geq N$ is also true. Finally, for the all x part, remember we had noticed $\frac{a^{N+1}}{1-a}$ is independent of x, then so is our error!

But, what to do with this meaningless conclusion. Fear not, since this conclusion is the definition of a very strong property called uniform convergence. What's that? Now, let's say $x = 1$. Take the sequence of numbers:

$$S_1(1), S_2(1), S_3(1), \ldots$$

If these numbers approach a limit, we say the functions converge at that point. Suppose when $x = 1$ we reach this true limit quickly but for $x = 0.1$ it takes much longer. So, there's no guarantee how the final function $W(x)$ might end up looking. While uniform convergence doesn't guarantee every point converges at the same speed but beyond some N, the approximation $S_N(x)$ will be uniformly close by ε to the final function $W(x)$ across the entire domain.

Bit vague? I know. Also how does any of this help with continuity? Let's think about it through graphics. First:

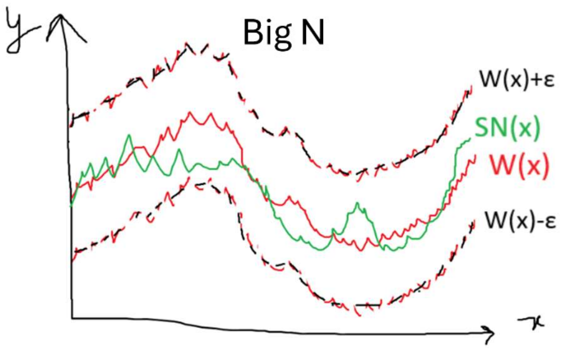
$$|S_N(x) - W(x)| < \varepsilon$$

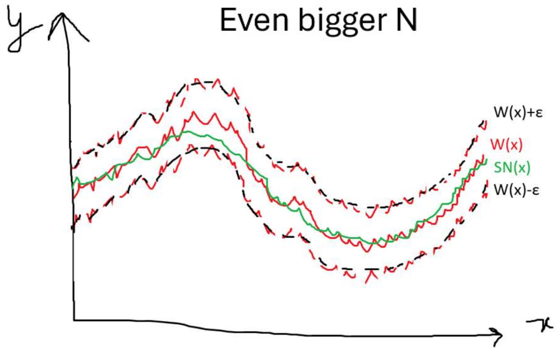
Upon rearranging becomes:

$$W(x) - \varepsilon < S_N(x) < W(x) + \varepsilon \tag{5}$$

Now what was $S_N(x)$? It is nothing but the finite sum of many special cos curves. We also know that regardless of how a cos curve is manipulated, it is always continuous (if you don't believe me https://www.desmos.com/calculator/ugazir94yu). And if we add 2 cos curves, as the curves themselves are continuous, then their sum should be too. Now if we do this summation $N + 1$ times we get $S_N(x)$ (given our special parameters), $\therefore$ $S_N(x)$ is continuous!

With this knowledge and (5) we know our estimate function is stuck between $W(x) \pm \varepsilon$, so we can make these graphs:





You'll have to excuse the MS paint, but the graph shows choosing a bigger value of N increases the squish of $W(x) \pm \varepsilon$ onto $S_N(x)$ because $\frac{a^{N+1}}{1-a}$ falls (shown in (2)). As we can make our error smaller than any chosen ε by taking N sufficiently large so $N \to \infty$, meaning, $S_N(x)$ eventually has no more free wiggle room and $S_N(x)$ uniformly converges to $W(x)$. Since $S_N(x)$ is continuous, $\therefore$ $W(x)$ is continuous!

Wow, that was a lot. OF FUN! Now, let's prove non-differentiability for a single point $x = 0$ as the proof for all points, is rather complex (Ramanujan's mentor, G.H Hardy proved it for all in 1916).

NON-DIFFERENTIABILITY PROOF AT $x = 0$

Using our definition of differentiability, if $W(x)$ is non-differentiable at $x = 0$, we need to show that:

$$\lim_{h \to 0} \frac{W(h) - W(0)}{h} \ does\ not\ exist \tag{6}$$

Rather than analysing all possible ways in which $h \to 0$, we focus on a specific sequence:

$$h_k = \frac{1}{b^k}$$

As $k \to \infty$, we have $h_k \to 0$, so this sequence ends up representing a valid way of approaching zero. If the limit fails to exist along this sequence, then the overall limit (6) cannot exist.

To do this, we substitute in $h_k = \frac{1}{b^k}$ and obtain:

$$g(k) = \frac{W(h_k) - W(0)}{h_k} = \sum_{n=0}^{\infty} a^n b^k (\cos(b^{n-k}\pi) - 1)$$

For convenience let $T_n = a^n b^k (\cos(b^{n-k}\pi) - 1)$. Once again, this infinite sum is difficult to deal with, so let's simplify life and split the sum as follows:

$$g(k) = \left(\textstyle\sum_{n=0}^{k-1} T_n\right) + \left(\textstyle\sum_{n=k}^{k} T_n\right) + \left(\textstyle\sum_{n=k+1}^{\infty} T_n\right) \tag{7}$$

Now, let's analyse each sum separately:

1. $\sum_{n=k}^{k} T_n \ (when\ n = k)$:

Here, $n - k = 0$ so $b^{n-k} = 1$. Simplifying the summation:

$$\sum_{n=k}^{k} a^n b^k (\cos(1 \cdot \pi) - 1)$$

$$= a^k b^k (-1 - 1)$$

$$= -2(ab)^k$$

2. $\sum_{n=k+1}^{\infty} T_n \ (when\ n > k)$:

Here, b^{n-k} is an odd integer, as b is an odd positive integer and an odd positive to the power of any positive integer ($n - k$ is a positive integer) is also odd. Hence:

$$\cos(b^{n-k}\pi) = \cos(odd \cdot \pi) = -1$$

Using this to simplify the summation:

$$\sum_{n=k+1}^{\infty} a^n b^k (\cos(b^{n-k}\pi) - 1) = b^k \sum_{n=k+1}^{\infty} a^n (-1 - 1) = -2b^k \sum_{n=k+1}^{\infty} a^n$$

Just like the continuity proof, if you notice $\sum_{n=k+1}^{\infty} a^n$ is a geometric series with the first term a^{k+1} and ratio a. So, using the formula, the sum to infinity is $\frac{a^{k+1}}{1-a}$. Hence, the summation is:

$$\sum_{n=k+1}^{\infty} T_n = \frac{-2a^{k+1}b^k}{1-a}$$

3. $\sum_{n=0}^{k-1} T_n \ (when\ n < k)$

Then $n-k<0$, hence:

$$b^{n-k} = \frac{1}{b^{k-n}} \ which\ is\ very\ small$$

As a result, there are no clean tricks we can opt for this time, but we can estimate our value down to a range. We can make use of the following inequality obtained through the Taylor series expansion (let's leave the derivation for this inequality for today) of cos:

$$|\cos(\theta) - 1| \leq \frac{\theta^2}{2} \ \forall\theta \in R$$

Now let $\theta = b^{n-k}\pi$, then:

$$\left|\cos(b^{n-k}\pi) - 1\right| \leq \frac{(b^{n-k}\pi)^2}{2}$$

Using this, we can say:

$$|T_n| = \left|a^n b^k\left(\cos\left(b^{n-k}\pi\right) - 1\right)\right| \leq \left|a^n b^k\right| \frac{(b^{n-k}\pi)^2}{2} = Ca^n b^{2n-k}$$

Where $C = \frac{\pi^2}{2}$ and conclude:

$$|T_n| \leq Ca^n b^{2n-k}$$

Or:

$$-Ca^n b^{2n-k} \leq T_n \leq Ca^n b^{2n-k}$$

Now we can consider our summation using the above inequality:

$$-1\sum_{n=0}^{k-1} Ca^n b^{2n-k} \leq \sum_{n=0}^{k-1} T_n \leq \sum_{n=0}^{k-1} Ca^n b^{2n-k}$$

Let's ignore the left-hand side for now as it's the same as the right-hand side multiplied by -1, so:

$$\sum_{n=0}^{k-1} T_n \leq \sum_{n=0}^{k-1} Ca^n b^{2n-k} = Cb^{-k}\sum_{n=0}^{k-1}(ab^2)^n$$

Once again we have reached a geometric series with finite sum till $k - 1\ term$ with the first term being 1 and ratio ab^2 giving $\sum_{n=0}^{k-1}(ab^2)^n = \frac{C(ab^2)^k - 1}{ab^2 - 1}$. Therefore, with simplification:

$$\frac{-C\left((ab)^k - b^{-k}\right)}{ab^2 - 1} \leq \sum_{n=0}^{k-1} T_n \leq \frac{C\left((ab)^k - b^{-k}\right)}{ab^2 - 1}$$

FINALLY, we can consider $g(k)$ from (7) not as a value but a range, so:

$$g(k) = \left(\sum_{n=0}^{k-1} T_n\right) + \left(\sum_{n=k}^{k} T_n\right) + \left(\sum_{n=k+1}^{\infty} T_n\right)$$

And now we know:

$$\frac{-C\left((ab)^k - b^{-k}\right)}{ab^2 - 1} - 2(ab)^k + \frac{-2a^{k+1}b^k}{1 - a} \leq g(x) \leq \frac{C\left((ab)^k - b^{-k}\right)}{ab^2 - 1} - 2(ab)^k + \frac{-2a^{k+1}b^k}{1 - a}$$

I will spare you the algebra, to get:

$$\lambda_1(ab)^k - \frac{\delta}{b^k} \leq g(x) \leq \lambda_2(ab)^k + \frac{\delta}{b^k}$$

Where $\lambda_1, \lambda_2\ and\ \delta$ are constants independent of k. As we want $k \to \infty$, then $(ab)^k \to \infty$ (as $ab > 1 + \frac{3}{2}\pi$) and $\frac{1}{b^k} \to 0$ (as $b > 1\ must\ be\ true$). Therefore, our inequality reduces to:

$$\infty - 0 \leq g(x) \leq \infty + 0$$

Or:

$$g(x) = \infty$$

Meaning when $x = 0$, $g(x) = \sum_{n=0}^{\infty} T_n$ diverges and hence $\lim_{h \to 0} \frac{W(h) - W(0)}{h}$ doesn't exist. Therefore at $x = 0$, the Weierstrass function is non-differentiable!

CONCLUSION

I guess the real smoothness was all the mathematical analysis we learnt along the way. Just kidding. Like our skins, math through gems like the Weierstrass function can teach a lesson on, what appears smooth at a glance can conceal intricate irregularities underneath. So, it is through PROOFS not perception, that we see what's truly present.