Finding a winning strategy for Nim

Richard Li

Introduction

I first came across Nim when playing a seemingly harmless, yet strategic game called “Pop It” on my friend’s phone. It sparked my curiosity and I ended up playing around 5 games with him, losing every single one. Salty, I went home and read up on how to win. I was actually playing “Nim”, a game played since ancient times. I came back the next day with the winning strategy to enact my revenge, winning every game. This ruined the fun and made him never want to play me again. Totally worth it.

In this essay, I hope to walk through the steps that might lead to finding a winning strategy for this game – something which Charles Bouton accomplished, whose 1901 paper[1] on the topic is regarded as having birthed the field of combinatorial game theory.

How to play

Nim is a simple two-player game played with multiple rows of objects, for example, coins. The number of objects in each row and the number of rows themselves does not matter, but the way to play is as follows:

- On a player’s turn, they select one row and can remove as many objects from that row as they want.

- The players switch turns after each move and the player who takes the last object wins

The important things are that on a player’s turn:

- They **must** remove at least one object

- Objects can only be removed from a single row on each turn

Example of a possible game

I will denote a state of a game of Nim by using numbers to represent the number of objects in each row and having the rows separated by commas. Note that the

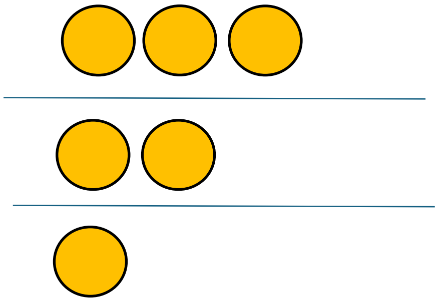
[1] https://paradise.caltech.edu/ist4/lectures/Bouton1901.pdf

order of the rows does not matter. The game (2,3,4) is the same as the game (4,2,3).

For example:

(3,2,1)

Means there are 3 rows with 3,2 and 1 objects in them respectively. The game will look something like this:



Now, suppose the two players are A and B respectively, with player A starting.

Player A may choose to remove 2 objects from the 1st row.

This results in the game now having state

(1,2,1)

Player B may now choose to remove all objects in row 2, resulting in the game with state

(1,1)

Player A is forced to choose a row with only one object in it and removes it.

(1)

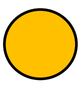
Player B chooses the last object and wins.

(0)

Now that you are familiar with the game of Nim, let’s start thinking like a mathematician.

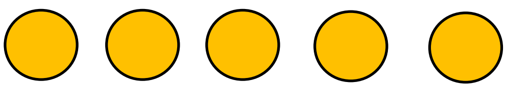
<u>One row</u>

A good place to start is with a single row containing one object. This game has a state of (1)



The player who starts will win as they can just take the only object on turn 1.

Likewise, if there is only one row, the game has state (n), where n is a natural number. The player who starts can win by simply choosing to take all n objects on their starting turn.

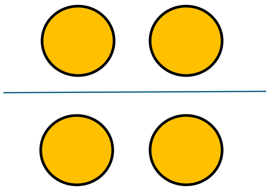


Two rows

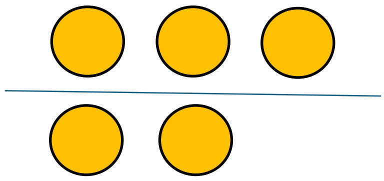
The next logical case to consider is with two rows.

Here are two scenarios with two rows/piles that are closely linked. Try to determine which player has a winning strategy and use that to deduce the winner in Scenario 2.

Scenario 1 (2,2)



Scenario 2 (3,2)



With some thought, we see that scenario 1 is winning for the second player or Player B.

This is because if player A chooses to remove all the objects from one row, then player B wins by taking all the objects on the other. So, player A tries to only remove one object. This leaves the game in state:

(2,1)

From there player B can mirror player A and remove one object from the row with 2 in it.

(1,1)

Player A is forced to remove one object.

(1)

Player B wins.

From this, we know that scenario 2 is winning for player A. They can force player B to start in scenario 1 by removing the "excess" object in the top row.

(3,2) Player A removes one object from row 1

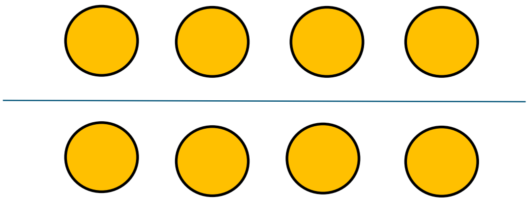
(2,2) Player B starts in scenario 1

…

Player B loses

Further exploration of the case with two rows leads us to imagine a third scenario with two rows of the same length e.g (4,4)

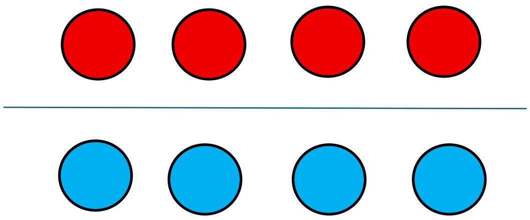
Try to determine which player has a winning strategy.



<u>Solution</u>

Player B has a winning strategy – and we can reason out why.

Let's colour the top and bottom rows red and blue respectively:



Since the two rows are identical, each red object has a corresponding blue object.

Whenever player A removes a number of red objects, player B can remove the same number of blue objects, and vice versa.

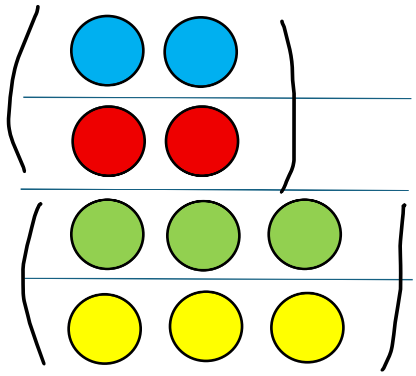
Doing this ensures that the red and blue rows will be identical every time after player B's turn.

Since the number of objects decreases each turn, player A is forced to remove the final object(s) from one of the colours, and then player B can remove the corresponding last objects of the other colour, winning the game.

In fact, we might realise that all games of Nim where there are an even number of each size of row can be won via this “copycat” strategy by player B. For example (2,2,3,3) is winning for player B.

This is because we can imagine grouping up the rows of the same size into individual pairs.

From here, player B can imagine colouring in the two identical rows different colours and proceeding as before for each imagined pair.



These (n,n) pairs do not change who is winning. You can just imagine they don’t exist at all and whenever the other player touches the rows, we proceed as above by copying, eventually forcing the pair to become (0).

This means we can just imagine the rows disappear or cancel each other out when we see (n,n) somewhere.

So, a big game like:

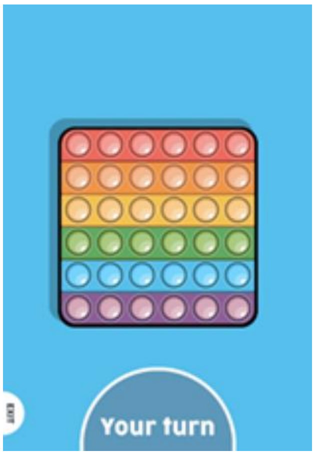
([8,8], [6,6], [5,5], [5,5])

Becomes

([0],[0],[0],[0])

Which just means that there is technically “nothing” left on the board as soon as the game starts. Hence why the player who starts loses in this case.

Now, this already brings us far enough to always win in “Pop It”, which was essentially just Nim with a 6x6 grid of objects (6,6,6,6,6,6). Here is a screenshot of the game:



(6,6,6,6,6,6) reduces to (0) and so is losing for the starting player.

You should politely invite your opponent to begin. That is also part of the strategy.

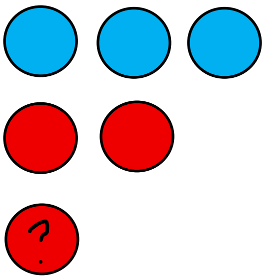
But we want to be as general as possible.

What happens when the game of Nim is chaotic with lots of piles of different sizes?

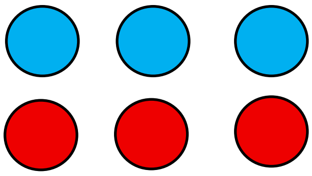
3 rows

After some experimentation with “chaotic” games with 3 piles, we may find that the game (3,2,1) is losing for player A. We also know that (3,3) is losing from our previous work.

(3, [2,1])



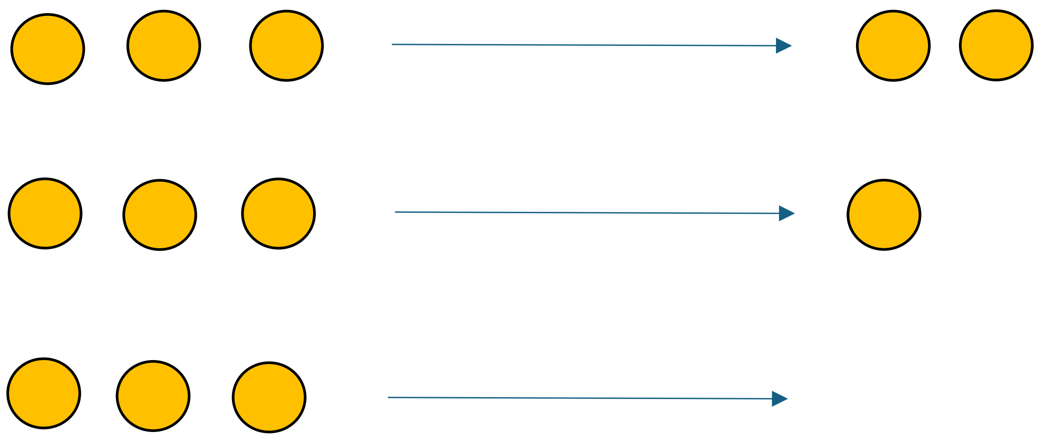
(3,3)



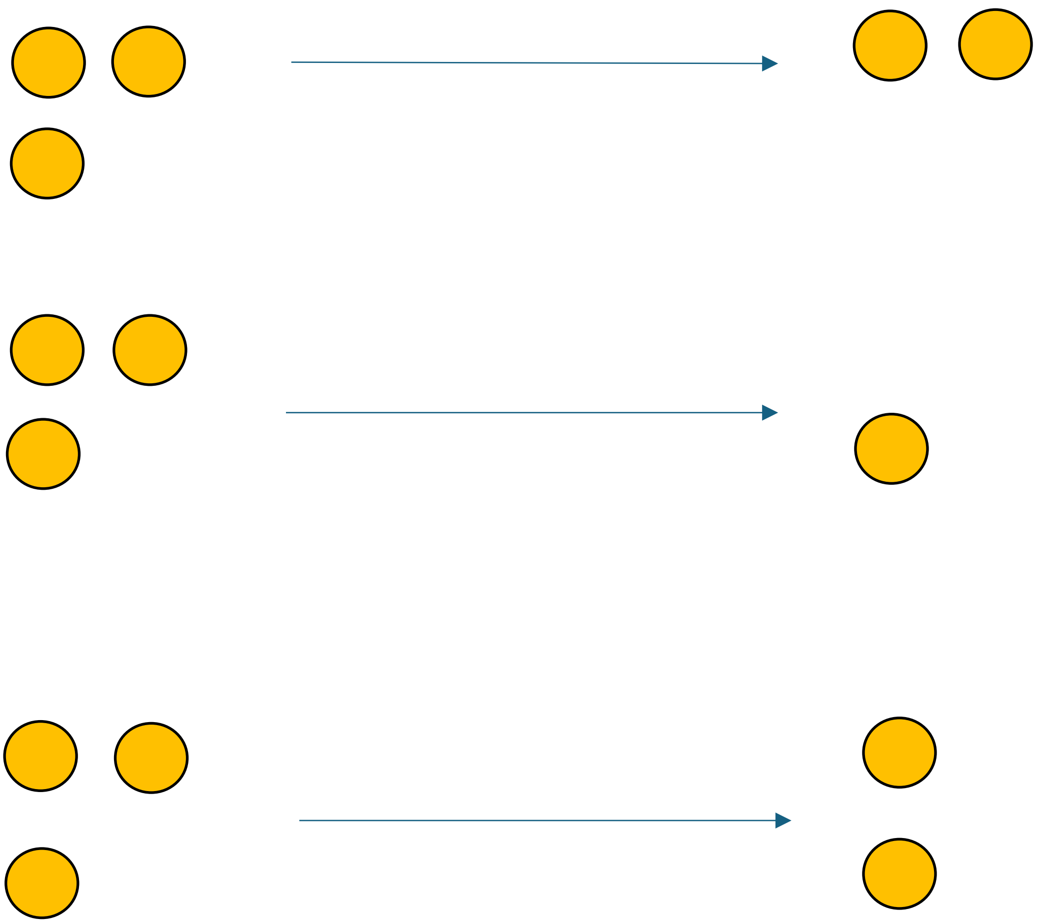
This is quite interesting and may lead us to the hypothesis that the red (2,1) is somehow behaving the same as the red (3)

Let’s think, what can you accomplish with a row of 3?

Well, there are only three things you can do, shown below:

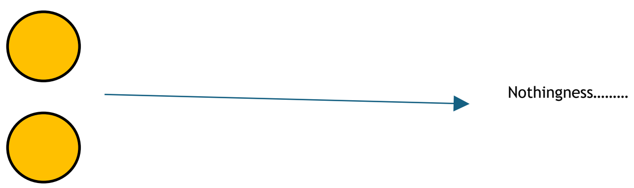


Now, let's consider what you can do with two rows (2,1).



But wait, we have already established that we can ignore (n,n), or treat it as (0)

Therefore, removing one object from (2,1) to create (1,1) effectively leaves (0)



This means that they both serve the same purpose and are interchangeable!

We can continue to find more sets that are interchangeable.

Let's define an operation $\oplus$ that adds together rows in our game of Nim and returns an equivalent position.

We know a few facts about $\oplus$

- $n \oplus 0 = n$ (2,0) is the same as (2)
- $n \oplus n = 0$ (n, n) cancels to leave (0)
- $A \oplus B = B \oplus A$ (2,1) is the same as (1,2)

Some interchangeable sets that we might have found with experimentation:

- $1 \oplus 2 = 3$
- $1 \oplus 4 = 5$
- $2 \oplus 4 = 6$
- $1 \oplus 2 \oplus 4 = 7$
- $6 \oplus 4 = 2$
- $2 \oplus 6 = 4$
- $2 \oplus 4 = 6$

With some mathematical intuition and luck, we can deduce this mystery operation.

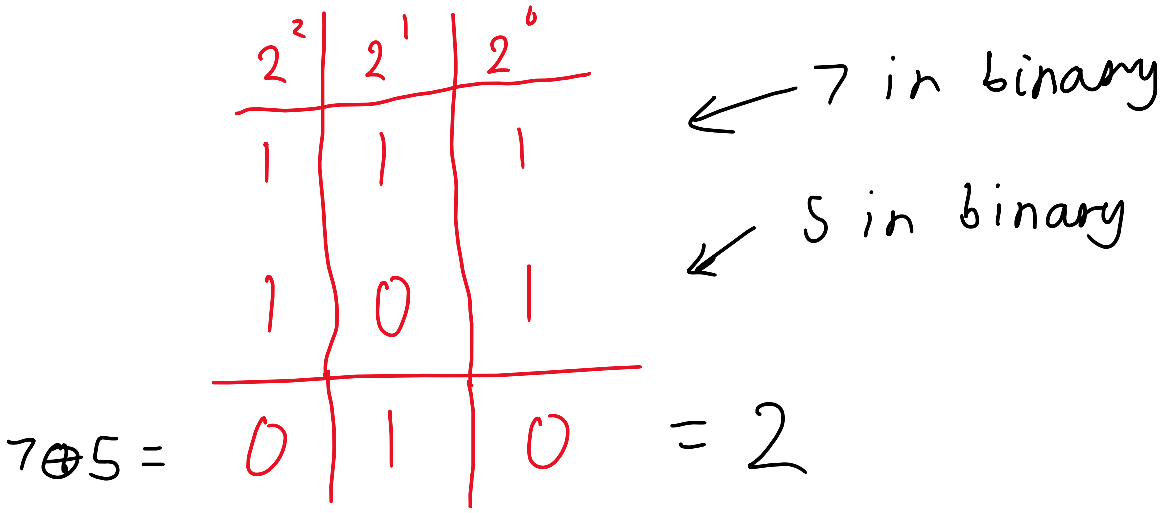
It seems, through our examples, that binary has something to do with Nim.

Looking at our last three examples and remembering that pairs of rows cancel out we might deduce that it is binary digits that are cancelling out.

This is exactly the Bitwise XOR.

How it works is we start by writing out the sizes of the piles in binary, column by column. If we have an even number of 1's in a binary column we write a 0 underneath, and a 1 otherwise.

For example, 7⊕5 would be calculated as follows:



How does this help us? Well, it means we can associate a game with a single number (N) which tells us if the position is winning or not.

Just as you might write 5+4+3+2+1=15, so too can you write a position of Nim as a single nim heap by adding all the nim heaps together using this special operator.

E.g. a game (5,4,3,2,1) can be represented by taking 5⊕4⊕3⊕2⊕1 = 1.

So, the game (5,4,3,2,1) is equivalent to the game with state (1)

Call the number N in this simplified state (N) the **nim sum.**

If the nim sum is 0, the position is losing.

How do we use this information to our advantage?

There are two things we need to show after a move:

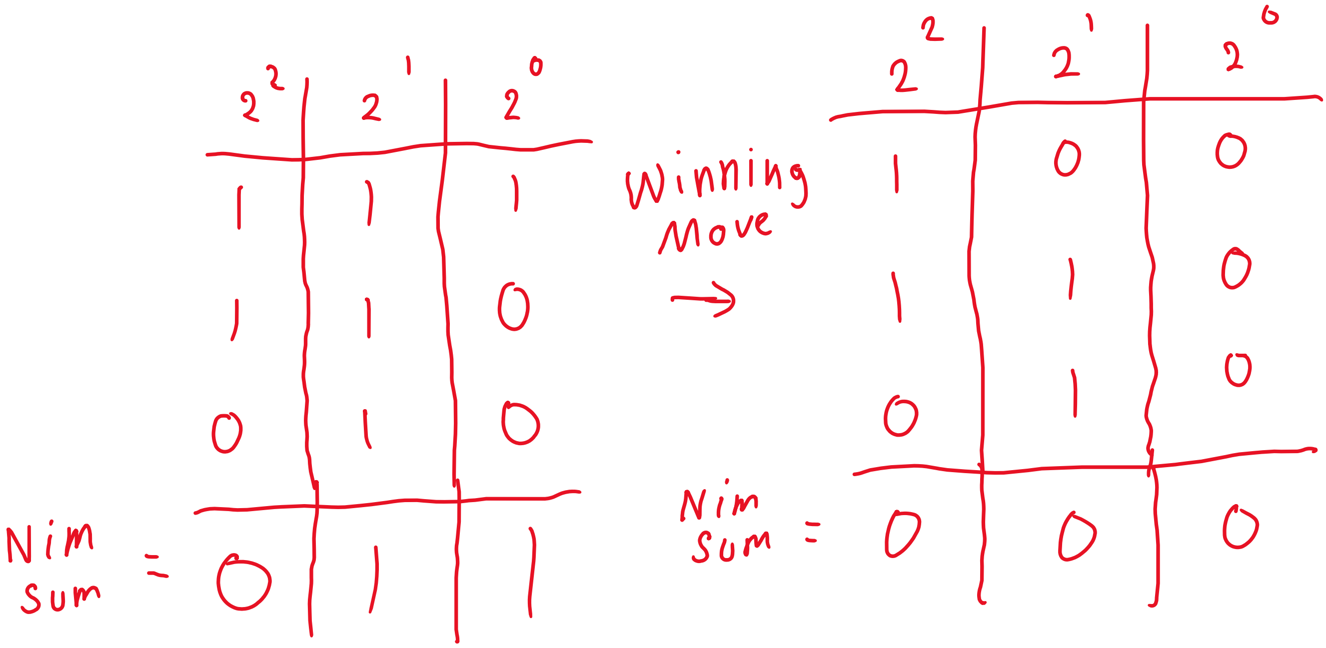
- It is always possible to change the nim sum of the game to 0 if it isn't 0
- It is guaranteed that the nim sum will become non-zero if it is 0

If we can prove these, then the winning strategy is to return the nim sum to 0 on every turn. Your opponent will only be handed positions with a nim sum of 0, and they have no choice but to change it. Since the number of objects decreases each move, they are eventually handed (0) meaning you have won!

Let’s try to see why these points are true:

<u>For point 1</u>

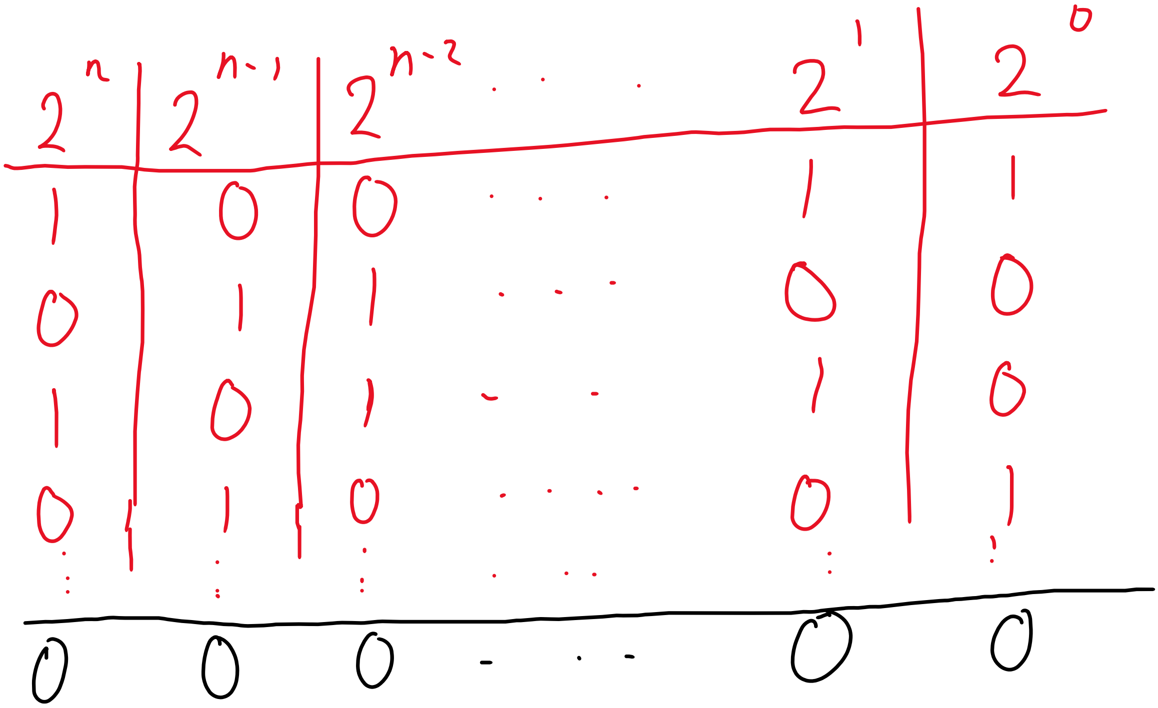
Here is the (7,6,2) game



Our goal – to get the nim sum to 0 - is the same as having an even number of 1’s in each column.

- We first identify the most significant power of 2 that has an odd number of 1’s. In this (7,6,2) case, 2^1.
- From there we remove objects from a row which has a 1 in that column, (one must exist as there is an odd number of 1’s in that column)
- Remove objects until that specific 1 flips to a 0. All lower value bits will then be 1.
- We can continue to remove objects such that all lower value bits end up with an even number of 1’s in each column.

For point 2



If the nim sum is 0, there is an even number of 1's in each column. Any move can only change the bits in one row, and it needs to flip at least one bit in that row. This ensures that the number of 1's will be odd in the changed column(s).

Conclusion

The winning strategy for Nim is:

- Return the nim sum to 0 on every move

Note that if the nim sum is 0 at the start of the game, you must hope your opponent makes a mistake, or you just let them begin.

What we have discovered may seem like an arbitrary party trick. However, it's actually the foundation of something called the Sprague-Grundy theorem. It proves that any strategic game (specifically any "impartial game") is equivalent to a game of nim with one pile, something like (n). [2]

Sometimes, a salty loss is all you need to kickstart a mathematical journey.

Thanks for reading.

[2] https://en.wikipedia.org/wiki/Sprague%E2%80%93Grundy_theorem

References

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https://play.google.com/store/apps/details?id=com.JindoBlu.TwoPlayerGamesChallenge&hl=en_GB

2. Nim, A Game with a Complete Mathematical Theory

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https://paradise.caltech.edu/ist4/lectures/Bouton1901.pdf

3. Sprague-Grundy theorem

https://en.wikipedia.org/wiki/Sprague%E2%80%93Grundy_theorem