The Pendulum and π; a unified story through elliptic integrals and the arithmetic-geometric mean

Fionn Murray

April 2026

1 Introduction

The simple pendulum is a canonical system in classical mechanics, yet its exact analysis reveals behaviour that lies beyond elementary methods. While small oscillations are well described by simple harmonic motion, with solutions expressed in terms of trigonometric functions, the full nonlinear equation

$$\ddot{\theta} + \frac{g}{l} \sin\theta = 0$$

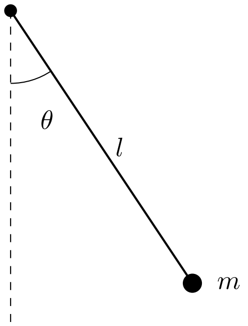
admits no solution in elementary form.

Instead, integrating the equations of motion leads naturally to elliptic integrals. These same integrals arise independently in classical geometry, most notably in the computation of the arc length of an ellipse, indicating that they capture a common underlying structure shared by seemingly distinct problems.

This essay develops the connection between nonlinear pendulum motion and elliptic integrals, before examining their analytical and computational properties. In particular, the arithmetic–geometric mean provides an efficient method for their evaluation, while Jacobi elliptic functions emerge as natural generalisations of trigonometric functions adapted to nonlinear systems.

2 The Pendulum

The simple pendulum is a fundamental model in classical mechanics, consisting of a point mass suspended from a fixed point by a rigid, massless rod under the influence of gravity. Despite its apparent simplicity, it exhibits qualitatively different behaviour depending on the amplitude of oscillation.



Pendulum of length l, bob mass m and displacement θ

Consider the height of the pendulum above the lowest point of the pendulum's swing:

$$h = l - lcos\theta = l(1 - cos\theta)$$

Gravitational potential energy is given as:

$$GPE = mgh = mgl(1 - cos\theta)$$

and the kinetic energy of the pendulum mass:

$$KE = \frac{1}{2}ml^2\dot{\theta}^2$$

And if released from rest at an initial displacement θ_0, conservation of mechanical energy gives:

$$\frac{1}{2}ml^2\dot{\theta}^2 + mgl\big(1 - \cos\theta\big) = mgl\big(1 - \cos\theta_0\big).$$

divide through by ml

$$\frac{1}{2}l\dot{\theta}^2 + g\big(1 - \cos\theta\big) = g\big(1 - \cos\theta_0\big).$$

subtract $g(1 - cos\theta)$ and simplify

$$\frac{1}{2}l\dot{\theta}^2 = g\big(1 - \cos\theta_0\big) - g(1 - cos\theta).$$

$$\frac{1}{2}l\dot{\theta}^2 = g\big(cos\theta - \cos\theta_0\big)$$

$$\dot{\theta}^2 = \frac{2g}{l}\big(cos\theta - \cos\theta_0\big)$$

Now differentiate both sides with respect to time:

$$2\dot{\theta}\ddot{\theta} = \frac{2g}{l}(-sin\theta)\dot{\theta}$$

After some simplification:

$$\ddot{\theta} + \frac{g}{l}sin\theta = 0$$

For small angular displacements, $\sin\theta \approx \theta$, so for an approximate solution for small amplitudes: we start by considering:

$$\ddot{\theta} + \frac{g}{l}\theta = 0$$

Assume a solution in the form: $\theta(t) = e^{rt}$

substituting this into our differential equation:

$$e^{rt}(r^2 + \frac{g}{l}) = 0$$

This gives the characteristic equation:

$$r^2 + \frac{g}{l} = 0$$

Solving for r gives

$$r = \pm\sqrt{\frac{g}{l}}i$$

Hence the general solution is

$$\theta(t) = Ae^{i\sqrt{\frac{g}{l}}\,t} + Be^{-i\sqrt{\frac{g}{l}}\,t}.$$

Apply the initial conditions

$$\theta(0) = \theta_0, \quad \dot{\theta}(0) = 0.$$

From $\theta(0)$:

$$\theta_0 = A + B.$$

Differentiate:

$$\dot{\theta}(t) = i\sqrt{gl}\,Ae^{i\sqrt{\frac{g}{l}}\,t} - i\sqrt{gl}\,Be^{-i\sqrt{\frac{g}{l}}\,t}.$$

From $\dot{\theta}(0) = 0$:

$$0 = i\sqrt{gl}(A - B) \;\Rightarrow\; A = B.$$

Hence

$$A = B = \frac{\theta_0}{2}.$$

So

$$\theta(t) = \frac{\theta_0}{2}\left(e^{i\sqrt{\frac{g}{l}}\,t} + e^{-i\sqrt{\frac{g}{l}}\,t}\right).$$

Using

$$\frac{e^{ix} + e^{-ix}}{2} = \cos x$$

We obtain

$$\theta(t) = \theta_0 \cos\bigl(\sqrt{\frac{g}{l}}\,t\bigr).$$

This approximate solution is elementary and admits a simple closed form in terms of trigonometric functions.

In the exact regime, the dynamics are no longer described by elementary functions. Instead, analysing the motion leads naturally to a new class of special functions, the elliptic integrals. Now consider the previous:

$$\dot{\theta}^2 = \frac{2g}{l}\bigl(cos\theta - \cos\theta_0\bigr)$$

isolating $\dot{\theta}$

$$\frac{d\theta}{dt} = \pm\sqrt{\frac{2g}{l}(\cos\theta - \cos\theta_0)}$$

and then a separation of variables and integrating both sides:

$$\int \frac{d\theta}{\sqrt{\cos\theta - \cos\theta_0}} = \int \pm\sqrt{\frac{2g}{l}}\,dt$$

Trigonometric identity. Using the half-angle formula

$$\cos\theta = 1 - 2\sin^2\left(\frac{\theta}{2}\right),$$

and similarly for θ_0, we obtain

$$\cos\theta - \cos\theta_0 = \bigl(1 - 2\sin^2(\frac{\theta}{2})\bigr) - \bigl(1 - 2\sin^2(\frac{\theta_0}{2})\bigr) = 2\left(\sin^2\left(\frac{\theta_0}{2}\right) - \sin^2\left(\frac{\theta}{2}\right)\right).$$

Substitute into the integral:

$$\int \frac{d\theta}{\sqrt{2\left(\sin^2(\frac{\theta_0}{2}) - \sin^2(\frac{\theta}{2})\right)}}.$$

Factor out $\sin^2(\frac{\theta_0}{2})$:

$$= \int \frac{d\theta}{\sqrt{2\sin^2(\frac{\theta_0}{2})\left(1 - \frac{\sin^2(\frac{\theta}{2})}{\sin^2(\frac{\theta_0}{2})}\right)}}.$$

Let

$$k = \sin\left(\frac{\theta_0}{2}\right).$$

Then

$$= \int \frac{d\theta}{\sqrt{2}\,k\sqrt{1 - \frac{\sin^2(\theta/2)}{k^2}}}.$$

Substitution. Let

$$\sin\left(\frac{\theta}{2}\right) = k\sin\phi.$$

Differentiate:

$$\frac{1}{2}\cos\left(\frac{\theta}{2}\right)\,d\theta = k\cos\phi\,d\phi \quad\Rightarrow\quad d\theta = \frac{2k\cos\phi}{\cos(\theta/2)}\,d\phi.$$

Using $\cos^2 x = 1 - \sin^2 x$, we get

$$\cos\left(\frac{\theta}{2}\right) = \sqrt{1 - \sin^2\left(\frac{\theta}{2}\right)} = \sqrt{1 - k^2\sin^2\phi}.$$

Substitute into the integral:

$$\int \frac{d\theta}{\sqrt{2}\,k\sqrt{1 - \frac{\sin^2(\theta/2)}{k^2}}} = \int \frac{2k\cos\phi}{\sqrt{2}\,k\sqrt{1 - k^2\sin^2\phi}\sqrt{1 - \sin^2\phi}}\,d\phi.$$

Simplify:

$$= \sqrt{2}\int \frac{d\phi}{\sqrt{1 - k^2\sin^2\phi}}.$$

Thus the integral reduces to

$$\int \frac{d\theta}{\sqrt{\cos\theta - \cos\theta_0}} = \sqrt{2}\,F(\phi, k),$$

Re-substituting our values for ϕ and k

$$= \sqrt{2}\,F\left(\arcsin\left(\frac{\sin(\theta/2)}{\sin(\theta_0/2)}\right),\ \sin\left(\frac{\theta_0}{2}\right)\right).$$

where

$$F(\phi, k) = \int \frac{d\phi}{\sqrt{1 - k^2\sin^2\phi}}$$

is the elliptic integral of the first kind.
Now we can solve for $\theta(t)$ from the reduced form,

$$F(\phi, k) = \pm\sqrt{\frac{g}{\ell}}\,t + C.$$

Apply $\theta(0) = \theta_0$, and since

$$\sin\frac{\theta}{2} = k\sin\phi, \quad k = \sin\frac{\theta_0}{2},$$

we have $\sin\phi(0) = 1$, hence $\phi(0) = \frac{\pi}{2}$, giving

$$C = F\Big(\frac{\pi}{2}, k\Big) = K(k),$$

where $K(k)$ is the complete elliptic integral of the first kind, i.e.

$$K(k) = F\Big(\frac{\pi}{2}, k\Big).$$

Thus

$$F(\phi, k) = K(k) \pm \sqrt{\frac{g}{\ell}}\, t.$$

Choose the negative branch (as $\dot\theta < 0$ for $t > 0$):

$$F(\phi, k) = K(k) - \sqrt{\frac{g}{\ell}}\, t.$$

The inverse of the elliptic integral of the first kind defines the *Jacobi amplitude*:

$$\phi = am(u, k) \quad where \quad u = F(\phi, k).$$

Thus

$$\phi(t) = am\Bigg(K(k) - \sqrt{\frac{g}{\ell}}\, t,\ k\Bigg).$$

The Jacobi elliptic sine function is defined by

$$sn(u, k) = \sin\big(am(u, k)\big).$$

Hence

$$\sin\frac{\theta(t)}{2} = k\,\sin\phi(t) = k\,sn\Bigg(K(k) - \sqrt{\frac{g}{\ell}}\, t,\ k\Bigg).$$

Therefore

$$\theta(t) = 2\ \arcsin\Bigg[k\,sn\Bigg(K(k) - \sqrt{\frac{g}{\ell}}\, t,\ k\Bigg)\Bigg], \quad k = \sin\frac{\theta_0}{2}.$$

The pendulum therefore provides a canonical example of how nonlinear systems in classical mechanics lead beyond elementary analysis into richer mathematical structures.

Furthermore, we can solve for the period of the pendulum, the time it takes for it to return to its initial displacement: Start from the implicit solution

$$F(\phi, k) = K(k) - \sqrt{\frac{g}{\ell}}\, t, \quad \sin\frac{\theta}{2} = k\sin\phi, \quad k = \sin\frac{\theta_0}{2}.$$

At $t = 0$, $\theta = \theta_0$, so $\phi = \frac{\pi}{2}$.
We now find the first time $t_1 > 0$ such that $\theta(t_1) = \theta_0$. This requires

$$\sin\phi(t_1) = 1 \;\Rightarrow\; \phi(t_1) = \frac{\pi}{2}.$$

From the periodic structure of the elliptic integral, the next occurrence corresponds to

$$F(\phi(t_1), k) = K(k) - 2K(k) = -K(k).$$

Hence

$$K(k) - \sqrt{\frac{g}{\ell}}\, t_1 = -K(k),$$

so

$$\sqrt{\frac{g}{\ell}}\, t_1 = 2K(k), \quad t_1 = 2\sqrt{\frac{\ell}{g}}\, K(k).$$

This is the time to return to the same angular displacement with opposite velocity (half an oscillation). Therefore the full period is

$$T = 2t_1 = 4\sqrt{\frac{\ell}{g}}\, K(k).$$

$$T = 4\sqrt{\frac{\ell}{g}}\, K\left(\sin\frac{\theta_0}{2}\right)$$

3 Gauss' Arithmetic–Geometric Mean and K(k)

Gauss' investigation of elliptic integrals of the first kind led to a striking and unexpected algebraic structure: he observed that certain transformations leave the value of the integral invariant, while systematically simplifying its parameters. This insight culminated in the arithmetic–geometric mean (AGM) iteration, in which two sequences converge rapidly to a common limit, encoding the value of the integral itself. The result is not merely theoretical elegance but computational power: the AGM provides an exceptionally fast method for evaluating elliptic integrals, and, by extension, constants such as π, with quadratic convergence. In this way, Gauss revealed a hidden unity between analysis and arithmetic—an instance where a seemingly intractable integral yields to a quiet, almost surgical iteration, exposing order beneath complexity.

We begin with the Legendre form of the complete elliptic integral of the first kind:

$$K(k) = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}.$$

Let

$$b = \sqrt{1-k^2}.$$

Then

$$1 - k^2\sin^2\theta = \cos^2\theta + b^2\sin^2\theta,$$

so define

$$I(a,b) = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{a^2\cos^2\theta + b^2\sin^2\theta}},$$

with

$$K(k) = I(1,b).$$

$$I(a,b) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{a^2\cos^2\theta + b^2\sin^2\theta}}.$$

Let $x = b\tan\theta$. Then

$$d\theta = \frac{b}{x^2+b^2}\,dx, \qquad \cos^2\theta = \frac{b^2}{x^2+b^2}, \qquad \sin^2\theta = \frac{x^2}{x^2+b^2}.$$

Hence

$$a^2\cos^2\theta + b^2\sin^2\theta = \frac{a^2b^2 + b^2x^2}{x^2+b^2} = \frac{b^2(x^2+a^2)}{x^2+b^2}.$$

So

$$\sqrt{a^2\cos^2\theta + b^2\sin^2\theta} = \frac{b\sqrt{x^2+a^2}}{\sqrt{x^2+b^2}}.$$

Therefore

$$I(a,b) = \int_0^\infty \frac{\frac{b}{x^2+b^2}\,dx}{\frac{b\sqrt{x^2+a^2}}{\sqrt{x^2+b^2}}} = \int_0^\infty \frac{dx}{\sqrt{(x^2+a^2)(x^2+b^2)}}.$$

Let

$$x = t + \sqrt{t^2 + ab}.$$

Then

$$\frac{dx}{dt} = 1 + \frac{t}{\sqrt{t^2 + ab}} = \frac{\sqrt{t^2 + ab} + t}{\sqrt{t^2 + ab}} = \frac{x}{\sqrt{t^2 + ab}},$$

so

$$dx = \frac{x}{\sqrt{t^2 + ab}}\, dt.$$

Compute x^2:

$$x^2 = (t + \sqrt{t^2 + ab})^2 = 2t^2 + ab + 2t\sqrt{t^2 + ab}.$$

Let

$$A = 2t^2 + ab + 2t\sqrt{t^2 + ab}.$$

Then

$$x^2 + a^2 = A + a^2, \qquad x^2 + b^2 = A + b^2,$$

so

$$(x^2 + a^2)(x^2 + b^2) = (A + a^2)(A + b^2) = A^2 + (a^2 + b^2)A + a^2b^2.$$

Now compute A^2:

$$A^2 = (2t^2 + ab)^2 + 4t^2(t^2 + ab) + 4t(2t^2 + ab)\sqrt{t^2 + ab}.$$

Expand:

$$(2t^2 + ab)^2 = 4t^4 + 4ab\,t^2 + a^2b^2,$$
$$4t^2(t^2 + ab) = 4t^4 + 4ab\,t^2.$$

Hence

$$A^2 = 8t^4 + 8ab\,t^2 + a^2b^2 + 4t(2t^2 + ab)\sqrt{t^2 + ab}.$$

Next,

$$(a^2 + b^2)A = (a^2 + b^2)(2t^2 + ab) + 2t(a^2 + b^2)\sqrt{t^2 + ab}.$$

Combine all terms:

$$(x^2 + a^2)(x^2 + b^2) = 8t^4 + (2a^2 + 8ab + 2b^2)t^2 + 2a^2b^2$$
$$+2t\big(4t^2 + a^2 + 2ab + b^2\big)\sqrt{t^2 + ab}.$$

Note

$$a^2 + 2ab + b^2 = (a + b)^2,$$

so

$$4t^2 + (a + b)^2 = 4\left(t^2 + \left(\frac{a + b}{2}\right)^2\right).$$

Thus

$$(x^2+a^2)(x^2+b^2) = 8t^4 + (2a^2+8ab+2b^2)t^2 + 2a^2b^2$$
$$+8t\left(t^2+\left(\frac{a+b}{2}\right)^2\right)\sqrt{t^2+ab}.$$

Now factor:

$$(x^2+a^2)(x^2+b^2) = 4\left(t^2+\left(\frac{a+b}{2}\right)^2\right)\left(2t^2+ab+2t\sqrt{t^2+ab}\right).$$

But

$$2t^2+ab+2t\sqrt{t^2+ab} = x^2,$$

so

$$(x^2+a^2)(x^2+b^2) = 4x^2\left(t^2+\left(\frac{a+b}{2}\right)^2\right).$$

Taking square roots:

$$\sqrt{(x^2+a^2)(x^2+b^2)} = 2x\sqrt{t^2+\left(\frac{a+b}{2}\right)^2}.$$

Substitute into the integral:

$$I = \int \frac{dx}{\sqrt{(x^2+a^2)(x^2+b^2)}} = \int \frac{\frac{x}{\sqrt{t^2+ab}}\,dt}{2x\sqrt{t^2+\left(\frac{a+b}{2}\right)^2}}.$$

Cancel x:

$$I = \int \frac{dt}{2\sqrt{t^2+ab}\,\sqrt{t^2+\left(\frac{a+b}{2}\right)^2}}.$$

Limits: as $x : 0 \to \infty$, $t : -\infty \to \infty$. Hence

$$I = \int_{-\infty}^{\infty} \frac{dt}{2\sqrt{(t^2+ab)\left(t^2+\left(\frac{a+b}{2}\right)^2\right)}}.$$

By even symmetry:

$$I(a,b) = \int_0^{\infty} \frac{dt}{\sqrt{(t^2+ab)\left(t^2+\left(\frac{a+b}{2}\right)^2\right)}}.$$

but note that t is just a "dummy" integration variable, it follows that:

$$I(a,b) = \int_0^{\infty} \frac{dx}{\sqrt{(x^2+ab)\left(x^2+\left(\frac{a+b}{2}\right)^2\right)}}.$$

recalling our previous expression for I(a,b):

$$I(a,b) = \int_0^\infty \frac{dx}{\sqrt{(x^2+a^2)(x^2+b^2)}}.$$

By comparing the two integrals, it becomes evident that

$$I(a,b) = I(\frac{a+b}{2}, \sqrt{ab})$$

we now define the sequences:

$$a_{n+1} = \frac{a_n + b_n}{2}, b_{n+1} = \sqrt{a_n b_n}$$

so we have

$$I(a,b) = I(a_1,b_1) = I(a_2,b_2) = ... = I(a_n,b_n)$$

Limit. Since $a_n, b_n \to AGM(a_0, b_0)$, we consider

$$I(a_n,b_n) = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{a_n^2\cos^2\theta + b_n^2\sin^2\theta}}.$$

note that $AGM(x,y)$ refers to the arithmetic-geometric mean of x and y For each fixed θ,

$$a_n^2\cos^2\theta + b_n^2\sin^2\theta \to AGM^2\sin^2\theta + AGM^2\cos^2\theta = AGM^2,$$

so the integrand converges pointwise to $1/AGM$.

Moreover, since $a_n, b_n > 0$ and bounded below,

$$\frac{1}{\sqrt{a_n^2\cos^2\theta + b_n^2\sin^2\theta}} \leq \frac{1}{\min(a_n, b_n)},$$

so the integrand is uniformly bounded. Hence, by dominated convergence,

$$\lim_{n\to\infty} I(a_n,b_n) = \int_0^{\frac{\pi}{2}} \frac{d\theta}{AGM} = \frac{\pi}{2AGM}.$$

Thus

$$I(a_0,b_0) = \frac{\pi}{2AGM(a_0,b_0)}.$$

Conclusion.

Since

$$K(k) = I(1, \sqrt{1-k^2}),$$

we obtain

$$K(k) = \frac{\pi}{2\,AGM(1, \sqrt{1-k^2})}.$$

linking this back to the period of the pendulum, recalling

$$T = 4\sqrt{\frac{\ell}{g}}\,K\left(\sin\frac{\theta_0}{2}\right)$$

using our new expression for $K(k)$, we get the extraordinary result:

$$T = 2\sqrt{\frac{l}{g}}\frac{\pi}{AGM(1, \cos\frac{\theta_0}{2})}$$

4 Closing remarks

The arithmetic–geometric mean transformation is remarkable both for its conceptual simplicity and its computational power. Starting from a seemingly intractable integral involving nested radicals, a carefully chosen sequence of substitutions reveals an unexpected invariance under the map

$$(a, b) \mapsto \left(\frac{a+b}{2}, \sqrt{ab}\right),$$

reducing the problem to the study of a pair of elementary iterative sequences.

What is striking is not merely that the transformation simplifies the integral, but that it does so *iteratively*: each step replaces the original parameters with ones that are closer together, while preserving the value of the integral. This process rapidly drives the pair (a_n, b_n) to their common limit, the arithmetic–geometric mean. The difficulty is not eliminated in a single step, but gently and systematically dismantled.

Moreover, the convergence is *quadratic*. Writing $d_n = a_n - b_n$, one finds

$$d_{n+1} \sim \frac{d_n^2}{4a_n},$$

so the number of correct digits roughly doubles at each iteration. What initially presents itself as an analytical obstacle is thus converted into a method of exceptional efficiency: accuracy compounds faster than intuition might first suggest.

At the same time, the transformation does not exist in isolation. The very same integral arises naturally in geometric and mechanical settings, such as the arc length of an ellipse and the period of a finite-amplitude pendulum. These are not artificial constructions but fundamental problems, and yet they lead directly to the same underlying structure. The AGM does not impose order upon them; rather, it reveals the order that was already there.

In this way, the AGM provides a bridge between analysis, geometry, and computation. A problem originating in measurement or motion is not merely solved, but reinterpreted as a rapidly convergent algorithm. The transformation shows that what appears complicated may in fact be highly structured, provided one looks at it from the right angle.

It is this synthesis that gives the AGM transformation its particular appeal. What begins as a problem in integration culminates in a method of striking efficiency, while simultaneously exposing a deep unity between seemingly disparate areas of mathematics. Simplicity, in this context, is not the absence of complexity, but its resolution.