Four Colour Theorem: A journey into Problem Solving

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1 Introduction

The Four Colour Theorem is one of those theorems that's really simple to state, but really hard to prove. In fact, the idea behind the problem is so simple that I found out about it while playing one of my favourite video games, *Persona 5 Royal*. Despite this, the theorem is notoriously difficult to prove, and that is what we will be looking at, as well as the insights proving this theorem gives us about the problem-solving process.

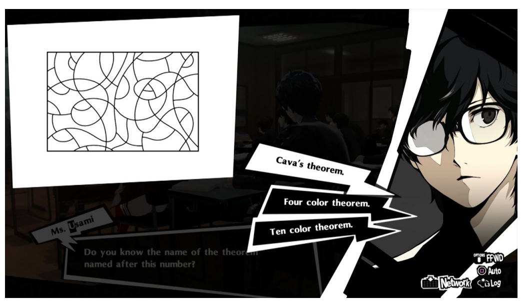
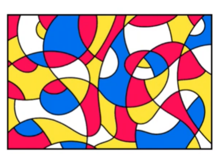


Figure 1: We have to answer the question correctly to raise our Knowledge stat. The other options are Ceva's theorem (spelled incorrectly), and "Ten Color theorem". The questions are meant to be easy to answer if you read the dialogue.

Essentially, as the game explains before giving the dialogue options, the Four Colour Theorem states that every region in any map can be coloured in a way that uses no more than four colours and has no two adjacent regions with the same colour. For example, this is the map from above coloured in this way:



2 Understanding the theorem

In order to prove this (or any theorem), we need to figure out what mathematical techniques we can employ, and understand the problem.
For the former, the current problem statement feels quite *qualitative*, and I highlighted some words in purple that especially need to be interpreted in a mathematical sense. To rectify this, mathematicians turn to Graph Theory, which lets us classify the problem nicely:

- Each region of the map, such as a country, is represented by a vertex.
- If two regions are adjacent, meaning they share a border, we represent this by connecting their two vertices with an edge.
- A map is an example of a planar graph. That is, it is theoretically possible to draw the graph without any edges overlapping, which must be true because in any map the borders of countries do not overlap. (Maybe this could happen if there was some kind of underground tunneling going on. Or with planes, I guess every country is adjacent.)
- Furthermore, maps are connected planar graphs, which means that there is always a path between any two vertices, and all vertices have at least one edge. If a map wasn't connected, it would mean that there was a country bordering literally nothing, which doesn't really make sense.

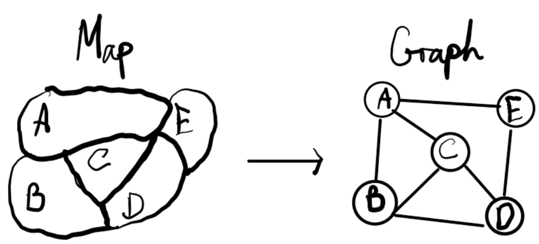


Figure 2: Representing a map as a graph. This graph is planar because we can draw it in such a way that no edges are overlapping.

This allows us to remove all unnecessary detail from the map, such as the shape or size of countries, and include only the details relevant to the problem. The theorem now becomes "it is possible to colour the vertices of a connected planar graph using no more than four colours such that no two vertices sharing an edge are of the same colour".

Now, there are two main questions that must be answered:

1. Why do we need at least four colours for any given map?

2. Why do we not need more than four colours for any map?

For the first question, refer back to the map I inserted and focus on the bottom right. Try to colour it with only 3 colours, and you will quickly run into issues (most likely because of that long curved yellow segment).
The second question is much more complicated, and is where the complexity of this theorem comes from. To begin to try to prove a statement like this, we need to think of an avenue to attack the problem, such as trying out easier versions of it like proving Ten Colour theorem. However, looking at Ten Colour Theorem would not be interesting, and I don't think anybody has ever wondered about it before, so we will instead look at a proof of Five Colour Theorem.

3 Proofs and Graph Theory

Before we tackle anything, we need to understand the maths that would let us prove any of these theorems. In fact, we need to understand what "proving" a theorem would even mean! The two main techniques in these proofs will be:

1. A proof by contradiction, where we assume that the theorem we are trying to prove is actually false, then showing that if the theorem is false, then something impossible must happen, and concluding the theorem is true.

2. A proof by induction, where essentially we show that if the theorem is true for some case (such as being true for a graph with n vertices; this assumption is called the inductive hypothesis) then it will be true for all larger cases (so we usually show true for n vertices $\Rightarrow$ true for $n+1$, and make a chain $n \Rightarrow n+1 \Rightarrow n+2 \Rightarrow n+3 \ldots$), and we then show that there is a small case where the theorem is true, the base case, so it is in fact true for all the larger cases.

Since we have quantified the problem as a result in Graph Theory, we can use ideas and results from graph theory to help us from now on. We start with some definitions:

1. We call the vertex representing country A v_A (a given vertex is v, and the subscript is just a label telling us it represents "A").

2. A path is a series of adjacent vertices that take you from one vertex to another. For example, we might take a path from v_a to v_c: $v_a \to v_b \to v_c$.

3. A cycle is a path which starts and ends at the same vertex, for example going from country $v_a \to v_b \to v_c \to v_a$, and no edges are used more than once (so we cannot go v_a to v_a, whatever that means, or from v_a to v_b to v_a).

4. The number of countries neighbouring country A is the degree of the vertex v_A , or $deg(v_A)$.

5. The total number of vertices in the graph is v, and the total number of edges is e
6. The number of *faces*, f, of a planar graph is the number of disjoint regions of a graph in its *planar representation* (the representation where none of the edges cross over)

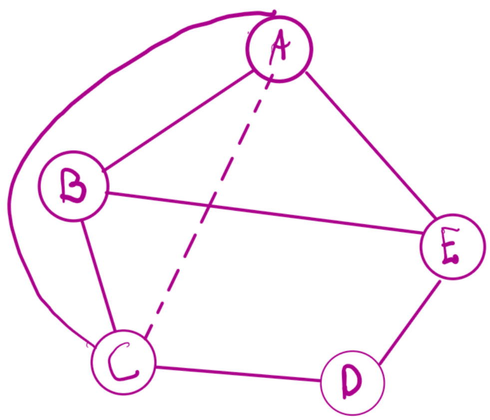


Figure 3: This graph is in its planar representation because we drew the edge from C to A so that it doesn't overlap the edge from B to E. For this graph, $v = 5$, $e = 7$, $deg(v_A) = 3$, $f = 4$.

Using these, we derive two crucial results from Graph Theory that will greatly aid us:

Lemma 3.1 (Handshake Lemma)**.** *If we sum the total number of neighbours of each country for every country, the total will be equal to twice the number of borders. Or,* $\sum deg(v) = 2e$

Proof. When we sum all the degrees, we are counting how many vertices A is connected to, B is connected to, etc. and so we count the total number of connections, but twice, as we count a connection from A to B and B to A separately. □

Lemma 3.2 (Euler's formula)**.** *For any connected planar graph,* $v - e + f = 2$

We use an inductive argument, where we show that we can start from a graph with $v - e + f = 2$ and construct any connected planar graph from this without changing the quantity, and so the constructed graph has $v - e + f = 2$. As we will often see from here on, the fact that the graph is planar, and thus has no overlapping edges is key and will be repeatedly used to put restrictions on the graph.

Proof. Consider the graph of two vertices connected by an edge. Because the graph in question is connected, we can always construct it by adding vertices to this graph, and so this will be the base case for induction, with

$$v - e + f = 2 - 1 + 1 = 2.$$

For the inductive hypothesis, assume that all graphs with n edges have $v - e + f = 2$. Consider adding an edge to a given graph.
The edge will either:

- Add a connection between existing nodes. e increases by one, and f increases by one, as a new face is enclosed, so $v - e + f$ does not change.
- Extend a path, adding a new node. v and e increase by one, $v - e + f$ does not change.

Either way $v - e + f$ is invariant; it does not change under the process of adding edges.
Hence, we can construct any connected planar graph with n edges with $v - e + f = 2$ by mathematical induction. □

4 Five Colour Theorem

We now look at how these results can help prove Five Colour Theorem.

Lemma 4.1. *In any connected planar graph, there exists a vertex with at most 5 adjacent vertices.*

Proof. Assume that some connected planar graph with a minimum degree of 6 exists. We count the total number of edges in terms of the total number of vertices in two ways (this idea is surprisingly known as Double Counting).

From the Handshake Lemma, $\sum deg(v) = 2e$.
Since the minimum degree of any vertex is 6, $2e \geq 6v \Rightarrow e \geq 3v$.

Now, consider relating the number of faces in the graph to the number of edges.

- Ignoring the outside face, for a face to exist, it must be enclosed by at least 3 edges (it may seem like curved lines can break this, but circles or semicircles or shapes with 2 or less edges require repeated edges between two nodes).
- Also, each edge creates a face on either side of it.
- So, $3f \leq 2e \Rightarrow f \leq \frac{2}{3}e$.

Substituting into Euler's Formula,

$$v - e + \frac{2}{3}e \geq 2$$
$$v \geq 2 + \frac{e}{3}$$
$$3v \geq e + 6$$

Combining with the Handshake Lemma, $e + 6 \leq 3v \leq e$, which is a contradiction.

□

Remark. Looking at extending this to Four Colour Theorem, it may be tempting to try and prove that the minimum degree of a connected planar graph is 4. However,we would eventually get by bounding e that $2.5v \leq e \leq 3v - 6$ which holds for $v \geq 12$.

Theorem 4.2 (Five Colour Theorem)**.** *For any connected planar graph, it is possible to colour the vertices using 5 distinct colours such that no two vertices sharing an edge have been coloured with the same colour. Such graphs are called five-colourable.*

Proof. Assume for induction that all graphs with $n-1$ vertices are five-colourable, and we now try to five-colour a graph with n vertices.
Consider the vertex v_{min}, with the smallest degree in the graph (this idea of looking at the smallest or largest object is known as the Extremal Principle).

- As shown above, the maximum degree v_{min} can have is 5.
- If we look at the graph while ignoring v_{min}, it has $n - 1$ vertices, so we can five-colour it from the inductive hypothesis.
- When we add v_{min} back in, if the adjacent vertices use less than 5 colours, then we can simply colour v_{min} a remaining colour, so this graph is five-colourable.
- Else, assume that v_{min} has degree 5, and has neighbours v_1, v_2, v_3, v_4, v_5 which are coloured with 5 separate colours, say red, purple, blue, pink, and green, respectively.

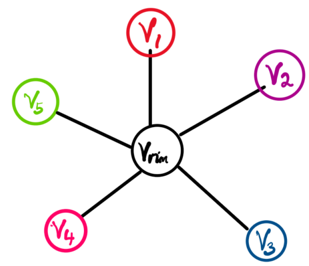


Figure 4: Trying to add v_{min} back to the graph, but the adjacent vertices use all five colours.

Consider the subgraph which contains only the purple and pink vertices, and edges between them.

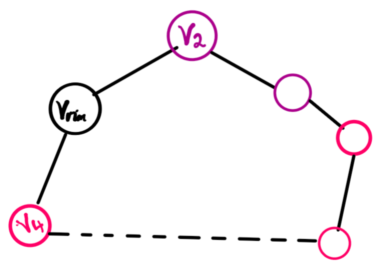


Figure 5: We look at a path from v_2 and see if it connects to v_4

If there does not exist a path between v_2 and v_4 in this subgraph, we can "invert" all of the vertices connected to v_2, switching the colour purple and pink. Note that this will not make any two adjacent vertices have the same colour.
Now, v_{min} is not adjacent to any purple vertices, meaning it can be coloured purple, and this graph is in fact five-colourable.

We repeat a similar argument for v_1 and v_3. The only way we won't five-colour the graph is if there also exists a path between v_1 and v_3.
In this case, the path between v_1 and v_3 must cross between v_2 and v_4, contradicting the planarity of the graph.
To complete the induction, the base case can just be a graph with two connected vertices (which is obviously five-colourable) as all connected planar graphs are made by adding edges and vertices to this graph.

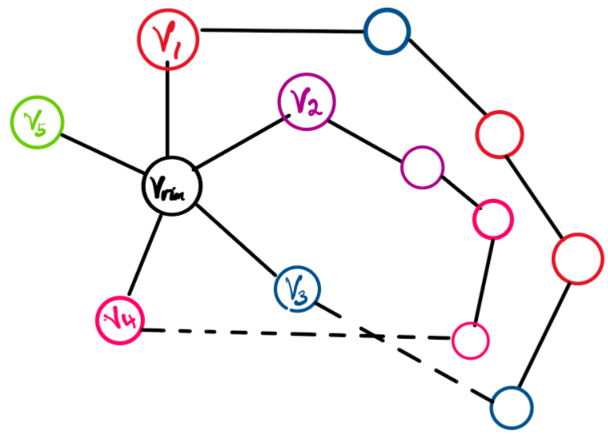


Figure 6: If there simultaneously exists paths between v_2 and v_4 and v_1 and v_3 they must cross, contradicting planarity

We conclude that by mathematical induction, all connected planar graphs are five-colourable. □

Remark. The main theme in this proof of making small disturbances in the graph and looking at small parts of the structure, as opposed to our previous proofs which look at the whole structure of the graph, which works really nicely here.
Remark 2. This does not directly extend to Four Colour Theorem, as even if we free up a colour in this way, the fifth vertex may have this colour. However, the idea of looking at these subgraphs (called Kempe chains) is involved in the proof of Four Colour Theorem.

5 Four Colour Theorem

If after reading these previous proofs, you're interested in how we extend these ideas for a proof of Four Colour Theorem, I have bad news for you.
Four Colour Theorem was actually the first theorem to be proved ”with a computer”, after only 124 years.
This is because after the clever techniques (which I have ran out of words to talk about), there were 1936 complex cases to check.

To conclude, the point of this was not to prove that we can four-colour graphs (I doubt anyone has genuinely wanted to do that, or needed the math if they did), but instead the journey that took us here. This all started with a rather basic question. For me, it was one I had to answer in *Persona* to raise my Knowledge stat. Then, this question lead to a journey of exploration, and the knowledge and tools acquired to understand Four Colour Theorem (such as the Extremal Principle, Invariants, proofs, and Double Counting) I argue are valuable in themself and make it all worthwhile.