Fractals – The ‘ematics’ in mathematics

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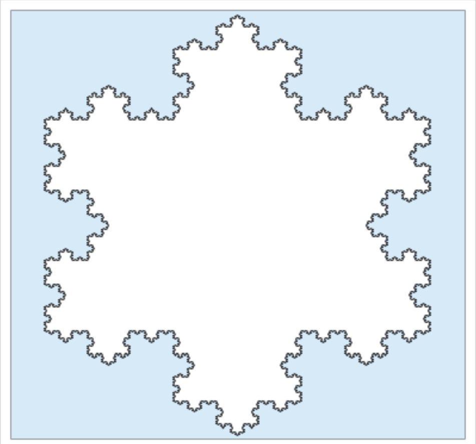
Introduction

You may be confused by the title of this essay, “The ‘ematics’ in mathematics”. Well, many people only see the ‘math’ in mathematics, the long equations, difficult problems and large computations. They fail to see the hidden beauty of mathematics; the patterns, structure and elegance, which I feel can best be represented through fractals.

Take for example the Koch Snowflake



Koch Snowflake

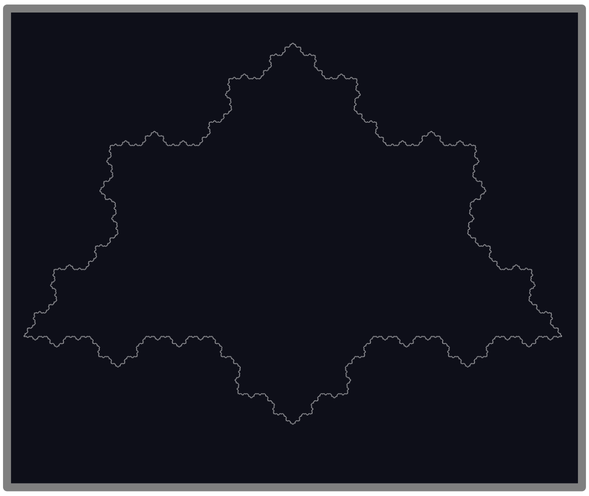
[Source: https://blogs.sas.com/content/iml/files/2016/12/koch1-300x300.png]

The steps to construct a Koch Snowflake are very simple:

- Take an equilateral triangle
- Divide each side into 3 equal parts
- Remove the middle third
- Replace the removed segment with two segments that form a small equilateral triangle bump outward.
- Repeat this process on every line segment infinitely.

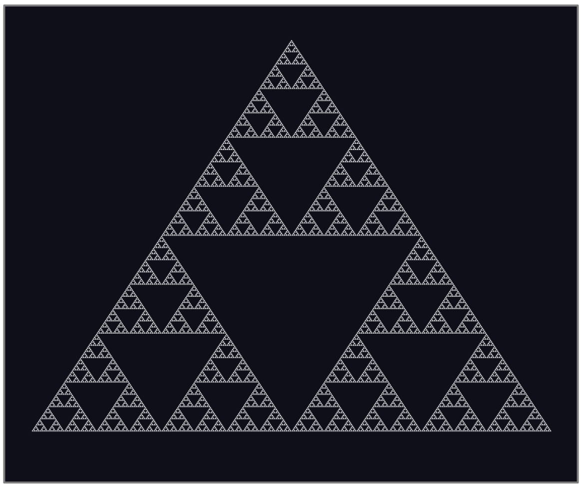
It’s amazing to think about how such simple steps can create so many elegant patterns.

Funnily enough, you can create a Koch Snowflake using isosceles triangles.
This Koch snowflake uses triangles with the central angle = 85 degrees!



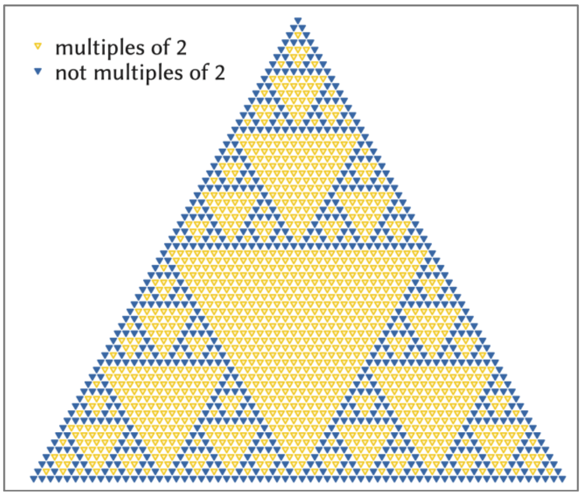
[Source: https://fractal-visualizer.netlify.app]

Another really interesting example of a fractal is the Sierpinski triangle:



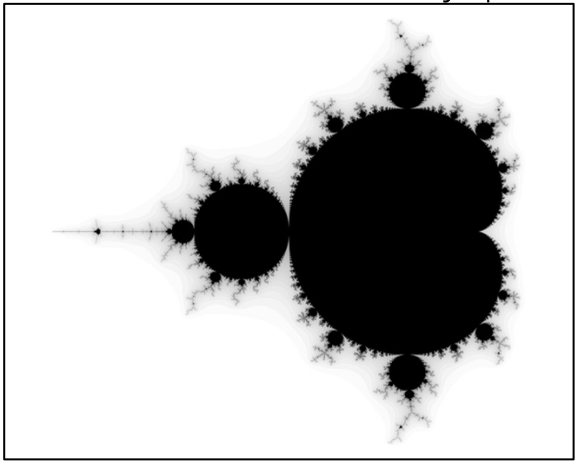
[Source: https://fractal-visualizer.netlify.app]

You might have noticed that it looks suspiciously similar to Pascal's Triangle with only odd numbers highlighted!



[Source: https://i0.wp.com/chalkdustmagazine.com/wp-content/uploads/2023/11/Screenshot-2023-11-23-at-19.03.11.png?resize=768%2C650&ssl=1]

The most well-known fractal in my opinion is the Mandelbrot set.



Source: https://paulbourke.net/fractals/mandelbrot/mandel1s.jpg]

It is represented by the equation:
$z_{n+1} = z_n^2 + c$
Where $c = a + bi$

All the elements of the Mandelbrot set are all values of c for which the sequence stays bounded. The x-axis represents the real part of c and the y-axis represents the imaginary part.

A dimension between 1 and 2…?

Euclid defined a point as having no dimensions, a line – being a breadthless length – having only one, and a plane surface – being a surface which lies evenly with the straight lines on itself – having two dimensions. However, some fractals have a dimension between 1 and 2!

This doesn't feel correct intuitively, having a dimension that isn't a whole number, yet this idea is essential in describing fractals, which do not fit neatly into classical geometric definitions.
A fractal dimension is represented by D

$$D = \frac{\log(N)}{\log(1/r)}$$

where N = number of self-similar pieces, and r = scale factor

This comes from $N = r^{-D}$

$$\rightarrow N = \frac{1}{r^D} \; or \; r^D = \frac{1}{N}$$

$$\rightarrow \log_r\left(\frac{1}{N}\right) = D \therefore \frac{\log\left(\frac{1}{N}\right)}{\log(r)} = D \rightarrow D = \frac{\log(N^{-1})}{\log(r)}$$

$$= -\frac{\log(N)}{\log(r)} = \frac{\log(N)}{-\log(r)} = \frac{\log(N)}{\log(r^{-1})} = \frac{\log(N)}{\log(1/r)}$$

But where does $N = r^{-D}$ come from?

Consider a line segment. Intuitively, we know that if a line is shrunk to half its original size, two such pieces.

Hence, N = 2, when $r = \frac{1}{2}$
Verifying the dimension of the line,

$$D = \frac{\log(N)}{\log(1/r)} = \frac{\log(2)}{\log\left(\frac{1}{1/2}\right)} = \frac{\log(2)}{\log(2)} = \log(2-2) = \log(0) = 1$$

Similarly, scaling the sides of a square to ½ of their original size, 4 of these squares would make one of the original square.

$$N = \left(\frac{1}{2}\right)^{-2} = 2^2 = 4$$

Scaling the sides of a cube to ½ of their original size, 8 of these cubes would make one of the original cube.

$N = \left(\frac{1}{2}\right)^{-3} = 2^3 = 8$

The dimensions of common fractals

Taking an example of the Koch Curve, each new segment is 1/3rd the size of the old one, hence $r = 1/3$.

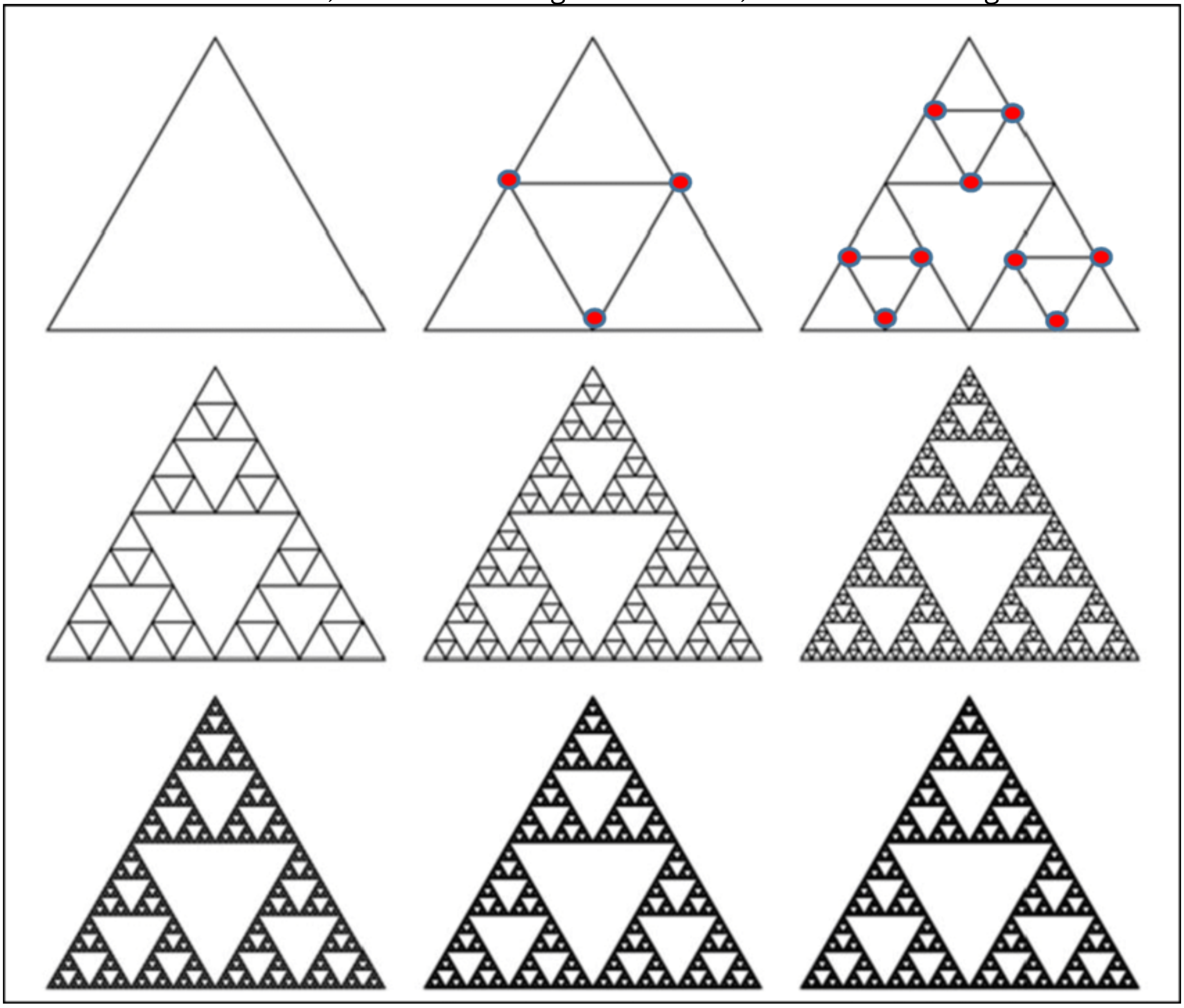
Through this, 4 new segments are formed, $N = 4$

Hence, the dimension of a Koch curve

$= D = \frac{\log(N)}{\log(1/r)} = \frac{\log(4)}{\log(3)} \approx \frac{0.602}{0.477} \approx 1.26$

In a Sierpinski triangle, each side of the triangle is scaled down to half of what it was and thus, 4 of these triangles are used to create one of the original triangle.

To turn this into a fractal, the middle triangle is removed, as shown in the figure.



[Source: https://www.researchgate.net/publication/359712336/figure/fig3/AS:1181185543806999@1658627987408/A-sample-of-the-Sierpinski-triangle.png]

Hence, in this case, $r = 1/2$ and $N = 3$, therefore, the dimension of a Sierpinski triangle is

$$D = \frac{\log(N)}{\log(1/r)} = \frac{\log(3)}{\log(2)} \approx \frac{0.477}{0.301} \approx 1.58$$

Notice that if the middle triangle isn't removed, the number of triangles formed is 4, hence $N = 4$. This would mean that the figure formed is essentially just a triangle, meaning it has 2 dimensions, being a flat shape. Verifying this:

$$D = \frac{\log(N)}{\log(1/r)} = \frac{\log(4)}{\log(2)} = \log_2 4 = 2$$

Infinite perimeter... but finite area?

In a Koch Snowflake, at each step, the perimeter is multiplied by 4/3.
Hence, the perimeter after the nth step

$$= P_n = P_0\left(\frac{4}{3}\right)^n$$

As the number of steps approaches infinity, the perimeter diverges to infinity
$\lim_{n\to\infty} P_n = \lim_{n\to\infty} P_0\left(\frac{4}{3}\right)^n$ and since $4/3 > 1$, $\lim_{n\to\infty} P_0\left(\frac{4}{3}\right)^n = \infty$

However, when it comes to the area, the series has to converge to a number, since it's an infinite sum of values that decrease exponentially. In this way, fractals like the Koch Snowflake have infinite perimeter, but finite area!

Are the fractals represented in pictures actually fractals?

Unfortunately not, when one sees a picture of a fractal like the Koch Snowflake, Sierpinski triangle, Mandelbrot set or the Candor Set, the image is only an example of how a fractal looks like.

Images of fractals are only approximations, because true fractals require infinitely many iterations and contain infinite detail, which cannot be fully displayed.

Conclusion

Although fractals may seem abstract, they are widely used in practical fields such as computer graphics, physics, and biology. However, I feel that fractals beautifully show the art and the abstractness of simple recursive math, truly being the 'ematics' in mathematics!