What makes two notes sound nice together? According to legend, Pythagoras may have formulated an answer to this when listening to the sound of blacksmiths' hammers. He realised that when certain hammers were struck on the anvil at the same time, if the ratio of the weights of the hammers was rational, like ½, the sound produced was pleasant[1]. While we now know that there may not be a demonstrable link between the hammer weights and the notes they produced, the theory behind it helps to build the maths behind Pythagorean musical tuning and cacophonous notes. We can also link the idea that rational frequency ratios produce harmonious sounds to the Lebesgue measure of rational numbers.

Sound is a wave – a form of energy transfer. It can be represented as a sine wave, with the frequency of the wave being the number of oscillations per second. When frequencies are high, we hear beats in quick succession, seeming to be a continuous note. However, when frequencies are very low, the note is not a continuous sound but a series of beats. Generally, when we layer sounds on top of each other, if the ratio of their frequencies are close to a "simple" rational number (meaning the denominator is small), the sound is pleasant. This may be due to a variety of reasons, be it evolutionary or psychological. Our ears find it easier to pick out rhythms and mathematical patterns when the frequencies (or beats) of the note have simple ratios, like 3/2 (a perfect fifth) rather than something more complex, like 37/23.[4] The exact frequency ratio does not need to be exactly equal to a rational number, however. A perfect fifth on a piano has a ratio of 2^(7/12), which is so close to 3/2 our ears (unless you were extremely musically trained) do not consider the ratio to be cacophonous.

Seemingly unrelated, measure theory seeks to assign a magnitude for the size for certain subsets of a larger set. The Lebesgue measure is a tool in measure theory. The idea behind the Lebesgue measure is that the size of the interval (a, b) should be equal to its length, b-a. The Lebesgue measure can also add disjoint intervals. For example, m((3,4) U (5,9)) = (4-3) + (9-5) = 5. In this case, m represents the Lebesgue measure. For these disjoint intervals (meaning they do not have an overlap), their total size is:

$$\sum_{k=1}^{n} m(a_k, b_k) = \sum_{k=1}^{n} (b_k - a_k)$$

And if we are trying to find the size of all the intervals, we can set the upper limit n = infinity.[2]

Using this information, we can say that:

$$\sum_{k=1}^{\infty} m(X_k)$$

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[1] Cichowlas, J., Dlotko, P., Kus, M. and Spalinski, J. (2025). *Consonance in music -- the Pythagorean approach revisited*. Available at: https://arxiv.org/pdf/2503.07632 (Accessed: 11 April 2026).

Where m(X) ≥ 0 for any measurable set X.[2] A set whose Lebesgue measure is well defined is called "measureable"[3]. Not every set is measurable, as some sets behave very irregularly and cannot be assigned a specific Lebesgue measure, like the Vitali set.

We can link these ideas to simplify the answer to a different question: how big are the rational numbers as a subset of real numbers? Imagine a number line, from 0 to 1. Is it possible to cover all rational numbers with intervals, with the total interval being less than 1? At first, this seems impossible. There are an infinite number of rational numbers and so it makes sense for their total length to be equal (or perhaps greater) than 1. However, counterintuitively, it is possible to prove that the total length of these intervals (their Lebesgue measure) is actually equal to 0.

First, we take all the rational numbers and order them. We use the fact that rational numbers are countably infinite, yet irrational numbers are uncountably infinite. This is due to Cantor's diagonal argument. If you listed all rational numbers (numbers that can be expressed as a fraction) diagonally, you would eventually work your way through all of them, which is not the same for irrational numbers (which cannot be expressed as a fraction). Returning to the original point, after enumerating all rational numbers, an interval is assigned to each of them. Because each number now has an interval, it now becomes slightly clearer that it is possible for the total sum of intervals to be less than 1. We can introduce a scaling factor ε, where 0 < ε (to show that the total interval must take up some space) and ε < 1 (since we are working within the limits of 0 and 1).

Choose any sum that equals 1. This type of sum does not diverge meaning the total of intervals will be finite and measureable. The simplest sum is:

$$\sum_{k=1}^{\infty} \frac{1}{2^k} = 1$$

It is important to note that this refers to the sum of all intervals, rather than the sum of all the rational numbers. The sum of all rational numbers does not converge, but the sum of the intervals (its Lebesgue measure) does.

Choose any value of ε, and multiply all the terms by it.

$$\sum_{k=1}^{\infty} \frac{\epsilon}{2^k} = \epsilon \sum_{k=1}^{\infty} \frac{1}{2^k} = \epsilon \times 1 = \epsilon$$

Since ε can be any number 0 < ε < 1, this means that the total sum of all intervals is ε. For any ε that we choose (where ε > 0) there are intervals that cover all rational numbers. The

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[2] Rochford, A. (2013) *There are Almost No Rational Numbers*, *Austin Rochford*. Available at: https://austinrochford.com/posts/2013-12-31-almost-no-rationals.html (Accessed: 11 April 2026).

[3] Tao, T. (2011). *An Introduction to Measure Theory*. Available at: https://terrytao.wordpress.com/wp-content/uploads/2012/12/gsm-126-tao5-measure-book.pdf (Accessed: 11 April 2026)

total length of those intervals is ε. The total length of the intervals cannot exceed ε. However, the measure of rationals is a single fixed number M. ε covers all rational numbers and M is the length of all rational number intervals. This means that $M \leq \varepsilon$. And as ε approaches 0 (meaning each interval becomes more precise by covering less space), M must always be slightly below it. While M could equal ε, it has to be below all possible values of ε, meaning M gets smaller and smaller until it reaches 0. That being said, it does seem paradoxical that there could be an infinite number of intervals, each covering a rational number between 0 and 1, with the total length of the intervals being 0.

Suppose the number line mentioned in the beginning actually represented different frequency ratios. If someone were to ask, “what frequencies sound pleasant on this line?”, without hearing the frequency one could simply point to the rational numbers (or irrational numbers that are close to rational numbers) to answer the question. If we were to cover all the rational numbers (pleasant-sounding ratios) with intervals, and set ε to 0.1, while all the rational numbers are covered by an interval, 90% of the number line is not. This means that 90% of the frequency ratios sound cacophonous.[4] ε, in this case, can be adjusted, getting smaller if the person's ear was more precise. Overall, this sets up an interesting link between a fundamental result of measure theory and seemingly unrelated real-world applications in music.

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[4] Sanderson, G. (4 Oct 2015) *Music And Measure Theory*, *YouTube*. Available at: https://www.youtube.com/watch?v=cyW5z-M2yzw (Accessed: 11 April 2026).