FROM COIN TOSSES TO ROCKET TRAJECTORIES: THE MATHEMATICS OF RANDOMNESS

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ABSTRACT

Randomness is often perceived as the absence of pattern or structure. However, mathematical analysis reveals that random systems are governed by well-defined probabilistic laws. This exploration examines randomness through probability, combinatorics and distribution theory extending further into algorithmic complexity and real-world aerospace systems. By analysing both theoretical models and applies scenarios, it is demonstrated that randomness isn't a disorder but a structured framework of multiple possible outcomes.

THE ILLUSION OF RANDOMNESS

Imagine being asked to write down the random sequence of ten-coin tosses: Heads(H) or tails (T). Most people will instinctively avoid repetition and produce a sequence like: HTTHTHHTHT.

Now consider this sequence: HHHHHHTTTT.

This appears suspiciously structured, almost as it were designed rather than generated by chance. Yet this instinct, compelling as it is, is mathematically unfounded and unsupported. Both sequences are equally likely.

For a fair coin, the probability of obtaining heads or tails on a single toss is ½. Because each toss is independent, the probability of any specific sequence of ten tosses is given by

$$\left(\frac{1}{2}\right)^{10} = \frac{1}{1024}$$

This result holds true universally and applies uniformly to all sequences. Randomness, therefore, cannot be determined by appearance; it is defined purely by probability.

COMBINATORICS AND APPARENT PATTERNS

While each sequence is equally likely, the number of sequences exhibiting certain characteristics varies significantly. The number of ways to obtain exactly r heads in the n tosses is given by

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

For n = 10, the number of sequences with exactly 5 heads is

$$\binom{10}{5} = 252$$

which is the maximum among all the possible values of r. By contrast, there is only

- one sequence with ten heads
- one with ten tails

Consequently, sequences with an approximately equal number of heads and tails occur far more frequently than the highly imbalanced one. Randomness, therefore, is not characterised by irregularity but by combinatorial dominance.

RANDOMNESS, COMPLEXITY AND EXPECTATION

Randomness can also be understood through algorithmic complexity. Sequences such as HHHHHHTTTT can be described by a simple rule whereas others like HTTHTHHTHT resist pattern and require longer descriptions. This idea, rooted in algorithmic information theory, suggests that randomness is linked to the difficulty of compressing a sequence. In this case, sequences that appear 'random' possess higher informational complexity. Thus, although all sequences are equally probable, they are not equally complex, revealing that intuition, while misleading in probability, captures an element of structure in a deeper mathematical sense.

This relationship becomes clearer when considering the repeated trials. If a coin is tossed n times, the expected number of heads is

$$E = np$$

For a fair coin, p = 1/2, so:

$$E = \frac{n}{2}$$

However, even for large n, the probability of obtaining exactly the expected value is relatively small. For example, when n = 100, the probability of obtaining exactly 50 heads is around 8%. The distribution instead spreads around the mean, producing variability that includes deviations and clusters. Such irregularities are not exceptions, but intrinsic features of random processes.

A common misconception arises when independence is overlooked. Each coin toss is unaffected by the previous outcomes, so

$$P\text{ (heads on the next toss)} = \frac{1}{2}$$

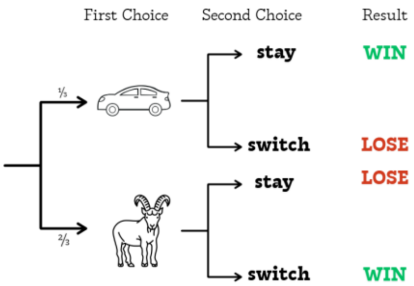
regardless of the prior results.

This directly contradicts the belief that outcomes must "balance out" in the short term. In reality, independence ensures that logical irregularities do not alter future possibilities.

INFORMATION AND DECISION: THE MONTY HALL PROBLEM

The role of probability becomes more subtle when information is introduced. This is illustrated by the Monty Hall Problem.

Figure 1: A probability tree illustrating the Monty Hall problem. If the initial choice is correct (probability 1/3), staying results in a win. If the initial choice is incorrect (probability 2/3), switching results in a win. Thus, switching yields an overall probability of success of 2/3.

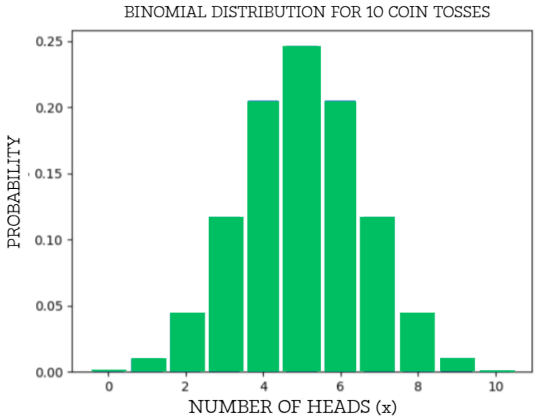


A participant initially selects one of the three doors with a probability of 1/3 of choosing correctly. The remaining two doors together carry the probability of 2/3. When one incorrect door is revealed, this 2/3 probability does not disappear, instead it transfers entirely to the remaining unopened door. Thus, staying gives a probability of success of 1/3 and switching gives a probability of success of 2/3. This result, though counterintuitive, demonstrates how probability is governed by structure rather than perception.

BINOMIAL DISTRIBUTION

This can be further visualised through the binomial distribution:

Figure 2: A graphical representation of the binomial distribution for the 10 coin tosses.



The graph shows that the probabilities peak at the centre and decrease symmetrically towards the extremes. This provides clear evidence that randomness produces structured patterns rather than complete unpredictability.

This structure can also be justified algebraically. Consider the expansion

$(1+1)^n$

By the binomial theorem,

$$(1+1)^n = \sum_{r=0}^{n} \binom{n}{r}$$

Since $(1+1)^n = 2^n$,

$$(2)^n = \sum_{r=0}^{n} \binom{n}{r}$$

Dividing both sides by 2^n

$$1 = \sum_{r=0}^{n} \binom{n}{r} \left(\frac{1}{2^n}\right)$$

This confirms that the binomial distribution accounts for all possible outcomes. Furthermore, the identity

$$\binom{n}{r} = \binom{n}{n-r}$$

explains the symmetry observed in the distribution.

RANDOMNESS IN REAL SYSTEMS: AEROSPACE APPLICATIONS

This mathematical understanding of randomness becomes significantly more compelling when considered in the context of aerospace systems, where the uncertainty is not theoretical, but unavoidable. In principle, the trajectory of a rocket can be described by deterministic equations of motion. However, in practice, the system is subject to numerous small uncertainties, including variations in atmospheric conditions, minor fluctuations in thrust, limitations in measurements, etc. Each of these factors introduces a random component into the motion of the spacecraft.

To model this mathematically, the ideal deterministic trajectory can be represented by a simple function such as y=x, which describes a perfectly predictable and linear path. However, to account for real-world uncertainties, a perturbation term must be introduced. This leads to a modified model of the form

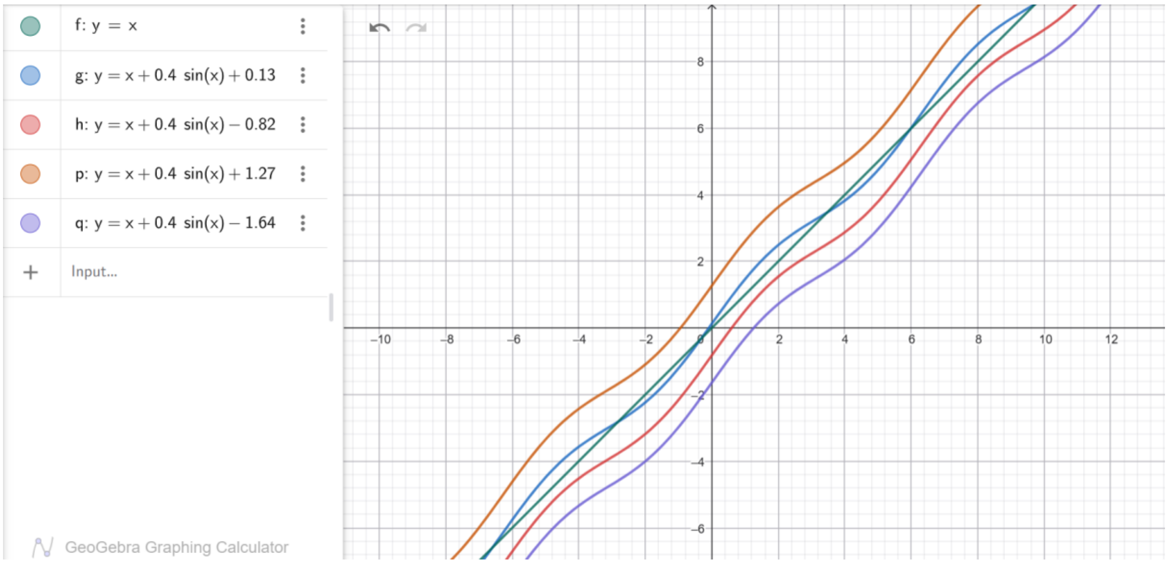
$$y = x + 0.4\sin x + \epsilon,$$

where ϵ represents a bounded random disturbance.

The sinusoidal component 0.4sinx reflects small systematic fluctuations, such as oscillations caused by the aerodynamic forces or minor instabilities in propulsion. The term ϵ, on the other hand, captures unpredictable variations arising from environmental and measurement uncertainties. Since these disturbances are not entirely unbounded in reality, ϵ is restricted within a fixed interval.

This behaviour is illustrated in the graph below, where multiple trajectories are plotted using different values of ϵ. Each curve represents a possible path that the rocket may follow under varying levels of uncertainty.

Figure 3: Multiple trajectories generated from the model y = x + 0.4sinx + ϵ, illustrating the effect of bounded random disturbances



f: $y = x$

g: $y = x + 0.4\ \sin(x) + 0.13$

h: $y = x + 0.4\ \sin(x) - 0.82$

p: $y = x + 0.4\ \sin(x) + 1.27$

q: $y = x + 0.4\ \sin(x) - 1.64$

It can be observed that although all the trajectories originate from the same deterministic model, they begin to diverge as the value of x increases. This demonstrates a key feature of probabilistic systems: small variations in initial conditions or external influences can result in significantly different outcomes over time.

Furthermore, the spread between the curves increases progressively, indicating that the uncertainty accumulates rather than remaining constant. This aligns with the real-world aerospace systems, where even negligible deviations at launch can propagate into significant divergences over long distances.

From this probabilistic perspective, each trajectory can be interpreted as a realisation of a random variable. The disturbance term ϵ can be considered to follow a uniform distribution within a fixed range, meaning that all values within the interval are equally likely.

As a result, the set of all possible trajectories forms a distribution rather than a single path. This reinforces that randomness is not the absence of structure, but the coexistence of multiple equally valid outcomes governed by probability.

This unifies the earlier examples of coin tosses and Monty Hall problem. In each case, the human intuition tends to misinterpret randomness by focusing on patterns or perceived fairness. However, as demonstrated mathematically, probability governs outcomes independently of how they appear. Similarly in aerospace systems, engineers cannot rely on intuition alone; instead, they must account for all possible variations using probabilistic models. The trajectory of a rocket is therefore not a single fixed path, but a range of possible outcomes, each with an associated likelihood.

CONCLUSION

Ultimately, the study of randomness reveals a profound and often counterintuitive truth: unpredictability does not imply disorder, but rather reflects an underlying mathematical structure that governs all possible outcomes. From coin tosses to combinatorial distributions and decision paradoxes, probability consistently demonstrates that what appears irregular is, in fact, rigorously determined. Yet, as seen through the lens of complexity, randomness is not merely about likelihood, but also about the absence of compressible structure, bridging gap between human intuition and mathematical reality.

This dual perspective becomes especially significant in real-world systems such as aerospace engineering, where deterministic models must coexist with unavoidable uncertainty. The introduction of even small perturbations transforms a single predictable trajectory into a distribution of possible outcomes, emphasizing that precision does not eliminate randomness but instead operates within it,

Therefore, randomness should not be viewed as a limitation, but as an essential feature of complex systems. By recognising that multiple outcomes can be equally valid, yet structurally distinct, we move beyond intuition and towards a more complete mathematical understanding of reality, one in which uncertainty is not an obstacle, but a framework through which the world can be more accurately described.