From Interest to Infinity: The Many Faces of *e*

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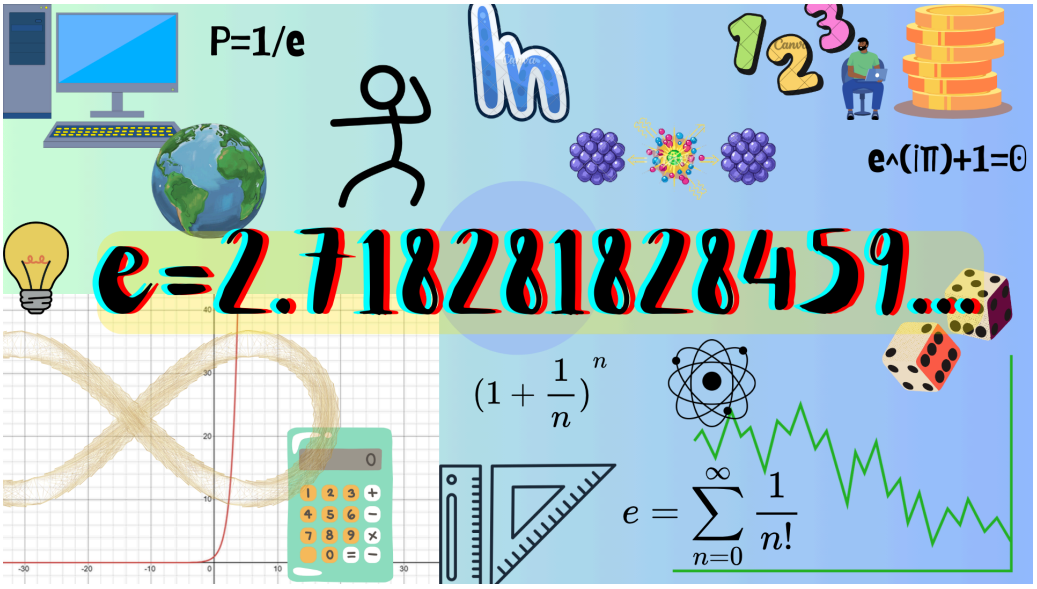


(Fig. 1: Cover image)

1. Introduction

Mathematics is beautiful, as I've been told. And I haven't yet stepped into the quiet, laboring, perennially melancholic nature that is characteristic to portrayed mathematical geniuses- no, I am a teenage girl, fond of nice and pretty things. If mathematics really *is* beautiful, this ought to be an enjoyable ride.

This essay has been written to satisfy a tiny curiosity of mine, for there is one thing that I see popping up in all kinds of unrelated places, which I know nothing about.

Let's imagine we are planting a seed. We water it everyday and hope it will grow. Each time it grows just a little, and over time it grows into something far bigger than imagined. Do numbers grow the same way? What happens if we repeat a tiny step again, and again, and again?

2. Let us compound

The birth of the number *e* is something well known and referenced, but I will talk about it again, because that's what rambling philosophers do best.

The 17th century was a time of rapid changes in trade, commerce, business and industries. In a period where mathematics was increasingly being used to study money, a natural question popped up when **Jacob Bernoulli**, a Swiss mathematician, was looking at problems about interest accumulating over time-
If interest is compounded more and more frequently, how large can the final amount get?
The formula to calculate total accumulated amount is

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

Suppose you invest $1 at 100% interest rate for one year.
Let us vary the value of n and see how much amount we get (note that we take P=$1, r=1, t= 1 year, and n represents the number of intervals in a year on which the compound interest is evaluated).

n	Value of $\left(1 + \frac{r}{n}\right)^{nt}$
1 (Annually)	2
2 (Semi-annually)	2.25
12 (Monthly)	$\approx$ 2.613
52 (Weekly)	$\approx$ 2.692
365 (Daily)	$\approx$ 2.714
When $\lim_{n \to \infty} \left(1 + \frac{r}{n}\right)^{nt}$	$\approx$ **2.71828**

He noticed something remarkable:

- The value keeps increasing
- But it doesn't grow without limit
- Instead it approaches a specific number

Bernoulli did not calculate the exact decimal value, but showcased that it was a constant lying between 2 and 3.

Do you notice something? Breaking growth into smaller and smaller steps makes the final result larger, but there is a limit.
There is a special number that can be seen as the largest value that results from splitting a fixed amount of growth into infinitely many tiny proportional

increases. It seems to represent a natural boundary between discrete and continuous growth.

The deeper significance of the number became clear in the work of the great mathematician Leonhard Euler (you may find it interesting that he was a student of Bernoulli's brother!). While the first constant used for this number was letter *b* by Leibniz in letters to Huygens in 1690, today we denote it by *e*, a symbol introduced by Euler.

3. Euler's number

Oddly enough, I have encountered the number e more in physics than mathematics. But I wanted to look at this number from a purely mathematical lens; from how it originated, to why it is so fundamental in seemingly all areas (from calculus, to radioactive decay rate, to computing). Can we draw a parallel between all these applications?

Mathematics often progresses in a funny but sublime way; ideas arise in different places, and only later is the hidden connection between them uncovered.

Most of Leonhard Euler's formal work on the number *e* has appeared in his 1748 book *Introductio in analysin infinitorum* ("**Introduction to the Analysis of the Infinite**").
Reading through Chapter VI (On Exponentials and Logarithms), we find-

1. Euler provides a description of Logarithms as "...This value of z, insofar as it is viewed as a function of y, is called the LOGARITHM of y... it has been customary to designate the logarithm of y by the symbol $log(y)$. If $a^z = y$, then $z = log_a y$." He also provides that the base of the logarithm must be a positive number $\neq 1$, and the argument must be greater than zero.
2. Some properties and methods of computing logarithms are discussed.

3. 1. Series Expansion

It is in Chapter VII (Exponentials and Logarithms Expressed Through Series) that we find the first description of the number *e*. Let us look at how Euler arrived at the first approximation of *e*.

We know that $a^0 = 1$. It follows that if the exponent is infinitely small and positive, the power exceeds 1 by a number infinitely small and positive. Let these numbers be of form-

$a^{\omega} = 1 + \psi$

We cannot say whether $\omega = \psi$, $\omega > \psi$, or $\omega < \psi$, for that depends on a variety of factors. But we can let $k\omega = \psi$. So,

$a^{\omega} = 1 + k\omega$

Raise both sides to a number j, which can be of any value we desire.

$a^{\omega j} = (1 + k\omega)^j$

Use binomial series to simplify it further-

$(1 + k\omega)^j = 1 + \frac{j}{1}k\omega + \frac{j(j-1)}{2!}k^2\omega^2 + \frac{j(j-1)(j-2)}{3!}k^3\omega^3 + \ldots$

Euler now takes a finite number z, and lets $j = \frac{z}{\omega}$. Because ω is infinitely small, $j \to \infty$. Substitute ω by $\frac{z}{j}$:

$a^z = 1 + kz + \frac{(j-1)}{2!}k^2\frac{z^2}{j} + \frac{(j-1)(j-2)}{3!}k^3\frac{z^3}{j^2} + \ldots$

"Since j is infinitely large, $\frac{j-1}{j} = 1$, and the larger the number we substitute for j, the closer its value comes to 1. Therefore, if j is larger than any assignable number $\frac{j-1}{j} = 1$,. For the same reason, $\frac{j-2}{j} = 1$, $\frac{j-3}{j} = 1$ and so forth. It follows that $\frac{j-1}{2j} = \frac{1}{2}$, $\frac{j-2}{3j} = \frac{1}{2}$, $\frac{j-3}{4j} = \frac{1}{4}$, and so forth." (Section 116)

This simplifies each term in the series.

- First term: 1, stays 1
- Second term: kz, remains same.
- Third term: $\frac{k^2z^2}{2!}$
- Fourth term: $\frac{k^3z^3}{3!}$

Hence, the series becomes-

$a^z = 1 + kz + \frac{k^2z^2}{2!} + \frac{k^3z^3}{3!} + \ldots$

This series expansion was used to provide an approximation of the number we wish to discuss.

3. 2. Definition and Notation

In section 122, Euler says "Since we are free chose base a for the system of logarithms, we now choose a in such a way that $k = 1$. Suppose now that $k = 1$, then the series found above in section 116,
$1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \ldots$ is equal to a. If the terms are represented as decimal fractions and summed, we obtain the value for $a = 2.718281828459045235360028\ldots$ When this base is chosen, the logarithms are called natural or hyperbolic… For the sake of brevity of this number… we will use the symbol e…"

It is said that Euler chose the symbol e as it was the next available vowel in his mathematical work after a, and perhaps because it is the first letter of "exponential".

It is quite fun to note that if you take Bernoulli's expression $\left(1 + \frac{x}{n}\right)^n$, expand it using Binomial's theorem and simplify by taking limit $n \to \infty$ (rewriting $\frac{n-1}{n} = 1$, $\frac{n-2}{n} = 1$,...) you get exactly the series expansion of e^x that Euler described!

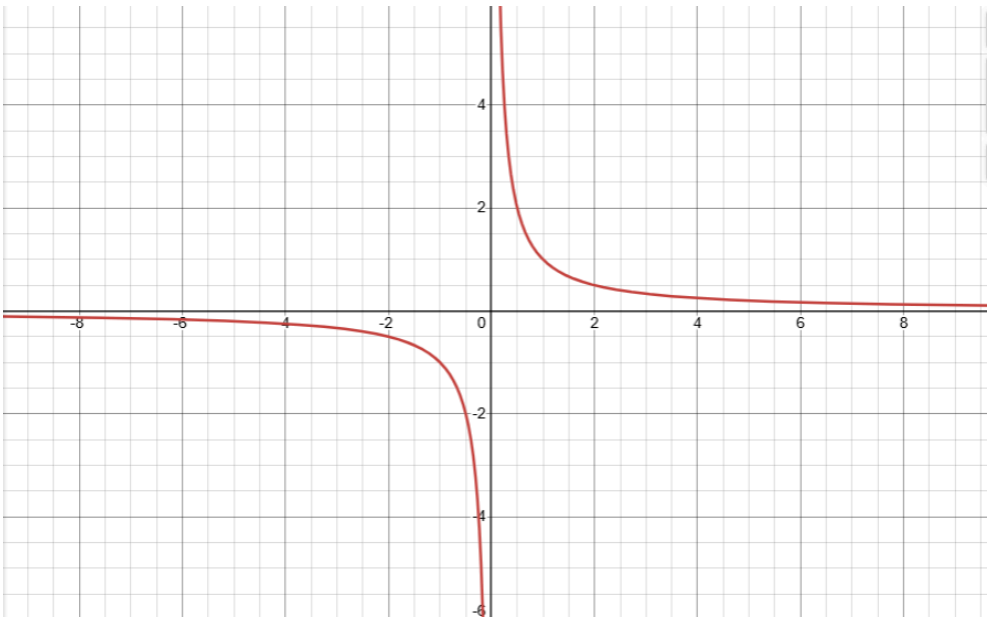
3. 2. 1. *Why "natural"?*

The term natural was not introduced by a single person at a single time, but rather developed through the 17th century. While no specific "justification" of the term can be found, it was likely because base e logarithms arise directly from the mathematics of exponential growth and series. As opposed to base 10 logarithms, which were chosen to simplify calculations, natural suggests something that arose from theory itself rather than being chosen for convenience.

3. 2. 2. *Why "hyperbolic"?*

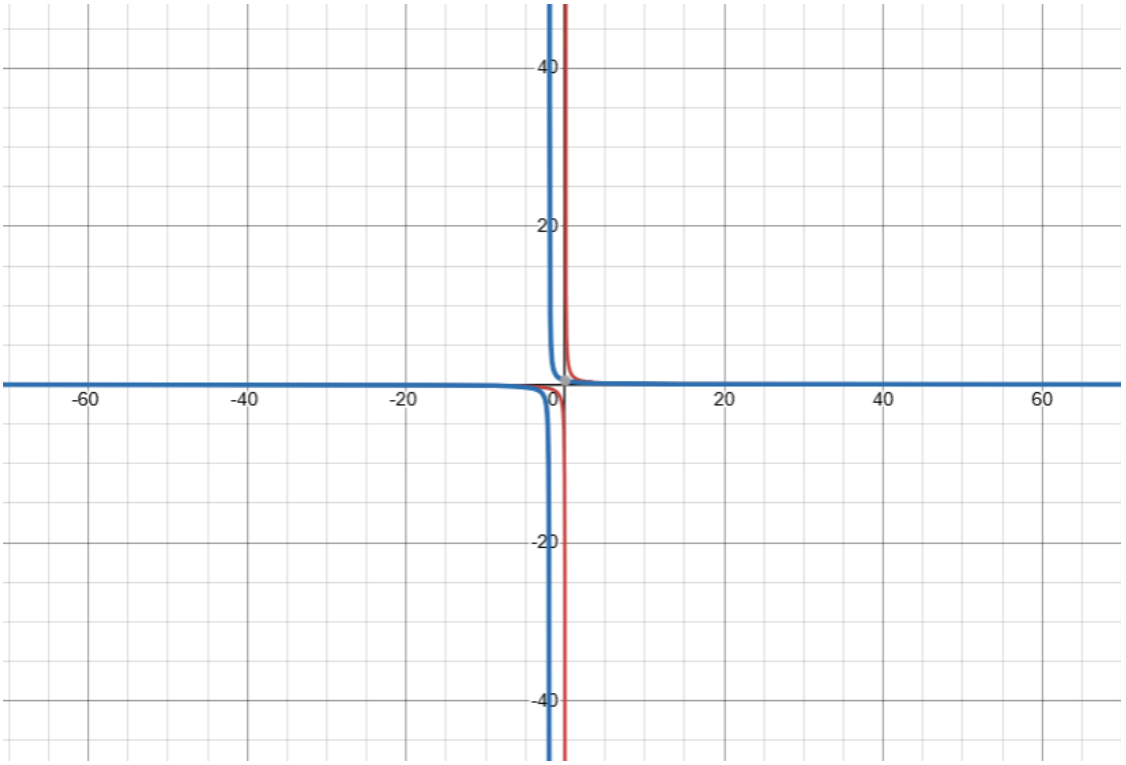
Because the quadrature of a hyperbola can be expressed by these logarithms. More precisely, suppose we find the area between $x = 1$ and $x = a$ for curve $y = \frac{1}{x}$.
The area is $\int_1^a \frac{1}{x} dx$. This integral evaluates to $ln(a)$.

4. Geometric Interpretation of e



(Fig. 2: Graph of hyperbola for equation $y = \frac{1}{x}$)

Let us increase x ever so slightly to $x + h$. Then, the extra area added is approximately in the form of a small rectangular strip as seen below-



(Fig. 3: Graph of hyperbola of equation $y = \frac{1}{x}$ (in red) and $y = \frac{1}{x+h}$ (in blue))

This extra area is approximately $\frac{1}{x} \cdot h$, since the width and height of strip are h and $\frac{1}{x}$ respectively.

Let $f(x) = ln(x)$. Then, $f(x + h) = ln(x + h)$. We can calculate derivative of $ln(x)$ w.r.t. x using first principle differentiation-

$$f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$$

(You will see that the numerator $f(x + h) - f(x)$ equals this extra area we described!)

$$f'(x) = \lim_{h \to 0} \frac{ln(\frac{x+h}{x})}{h}$$

$$f'(x) = \lim_{h \to 0} ln(\frac{x+h}{x})^{\frac{1}{h}}$$

Let $\delta = \frac{h}{x}$, and as $h \to 0,\ \delta \to 0$.

$$f'(x) = \lim_{\delta \to 0} ln(1 + \delta)^{\frac{1}{\delta x}}$$

$$f'(x) = \frac{1}{x} \lim_{\delta \to 0} ln(1 + \delta)^{\frac{1}{\delta}}$$

Let $\delta = \frac{1}{n}$. As $\delta \to 0,\ n \to \infty$

$$f'(x) = \frac{1}{x} \lim_{n \to \infty} ln(1 + \frac{1}{n})^{n}$$

$$f'(x) = \frac{1}{x} ln\{ \lim_{n \to \infty} (1 + \frac{1}{n})^{n} \}$$

Remember Bernoulli's expression for e in the context of compound interest?

$$f'(x) = \frac{1}{x} ln(e)$$

$$f'(x) = \frac{1}{x}$$

Or $ln(x) + C = \int \frac{1}{x}\, dx$

We know $ln(1) = 0$ ($e^0 = 1$)

Substitute $x = 1$

$$ln(1) + C = \int_1^1 \frac{1}{x}\, dx$$

$C = 0$ (Area from 1 to 1 is zero)

So, $ln(x) = \int \frac{1}{x}\, dx$

The area under curve $y = \frac{1}{x}$ from 1 to some number a is:

$\int_1^a \frac{1}{x}\,dx$

Therefore, $\int_1^a \frac{1}{x}\,dx = ln(a) - ln(1)$

$\int_1^a \frac{1}{x}\,dx = ln(a)$

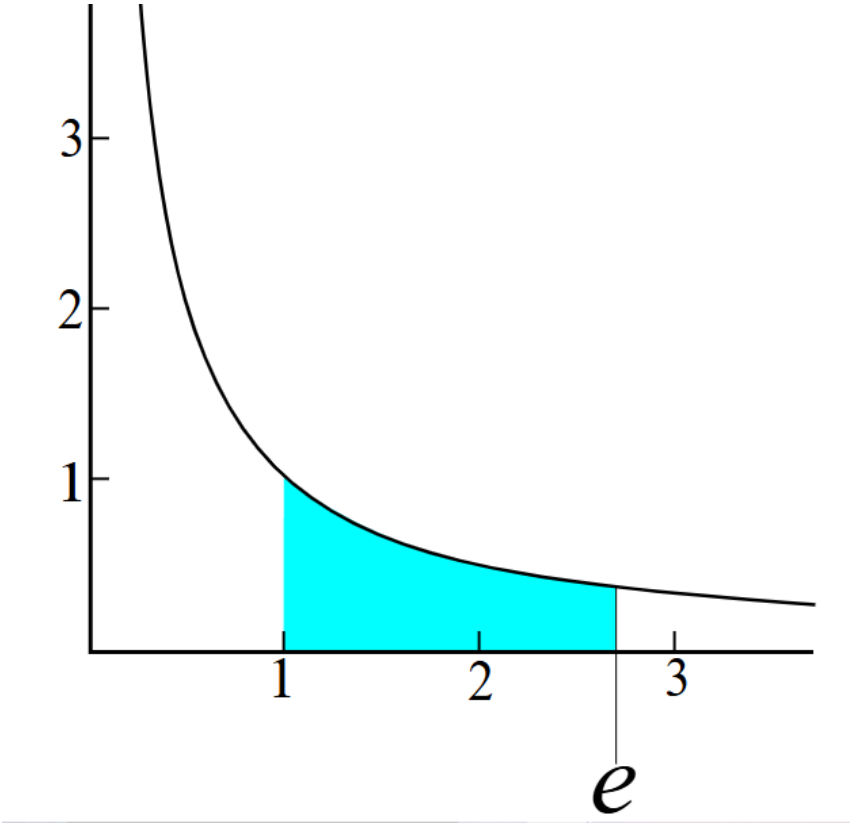
If we choose $a = e$

$\int_1^e \frac{1}{x}\,dx = 1$

This suggests that e is the unique number for which the area under $y = \frac{1}{x}$ between $x = 1$ and $x = e$ is exactly 1 square unit. e is the number where the accumulated rate $\frac{1}{x}$ reaches exactly 1.

Also, the fact that $ln(a)$ is the total accumulated value of $\frac{1}{x}$ from $x = 1$ and $x = a$ tells us that logarithms are not only algebraic tools but measure accumulated change along a curve.

Perhaps that is why they appear in many natural processes involving rates that decrease like $\frac{1}{x}$!



(Fig. 4: Graph of the equation $y = 1/x$ where e is the unique number larger than 1 that makes the shaded area under the curve equal to 1: By Cronholm144 at English Wikipedia, CC BY-SA 3.0, https://commons.wikimedia.org/w/index.php?curid=3128774)

5. Applications of this result

Let $y = ln(x)$

Then, $x = e^y$

Differentiate both sides w.r.t. x

$\frac{dy}{dx} = \frac{d(ln(x))}{dx}$

$\frac{dy}{dx} = \frac{1}{x}$

$\frac{dy}{dx} = \frac{1}{e^y}$ $\quad (x = e^y)$

$\frac{dx}{dy} = e^y$

But $x = e^y$

$\frac{de^y}{dy} = e^y$

In its more general form, $\frac{de^x}{dx} = e^x$. The fact that e^x is its own derivative tells us that the rate of change of the function is exactly equal to the function itself.

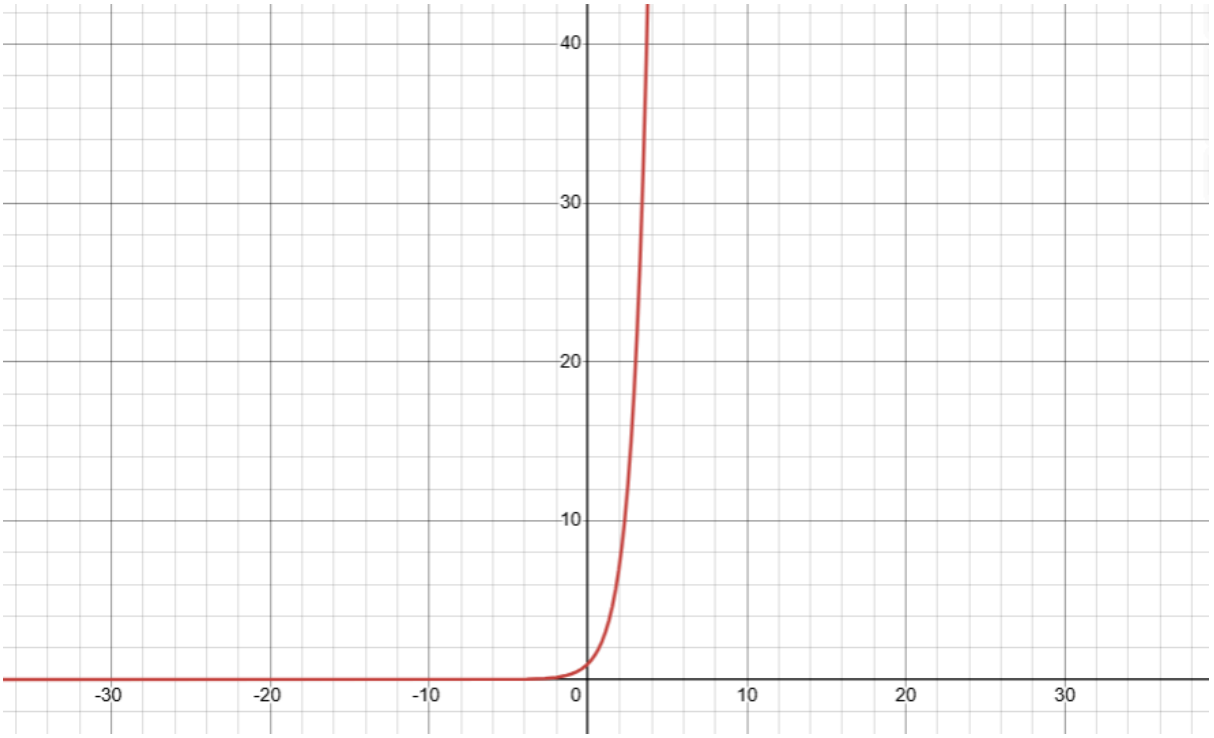
$\frac{dy}{dx} = y$ is one of the simplest differential equations. Its solution is Ce^x.

It means the faster something grows, the more it grows, and its growth rate is proportional to its current size. There are many places where $rate\ of\ change \propto amount\ present$:

- Population growth
- Compound interest
- Radioactive decay
- Spread of diseases
- Cooling and heating processes

The exact specifics would a long story for another time, but here we attempted to explain why e is seemingly used in such diverse applications in our day-to-day lives.

For a general exponential, $\frac{da^x}{dx} = a^x ln(a)$. So normally, the derivative introduces an extra constant. But in case of e, $ln(e) = 1$, so the derivative becomes cleaner and simpler, which was useful in calculations in historical times. That is why it is considered a "natural" base and shows up in physics, biology, economics and probability- whenever rates of change are involved.



(Fig. 5: Graph of $y = e^x$)

One of the defining features of natural logarithms mentioned by Euler was that for very small increments, the logarithm equals the increment itself. ("Natural logarithms have the property that the logarithm of (1+ ω) is equal to ω, where ω is an infinitely small quantity.")
This is another reason why they were called "natural"- they behaved in the simplest way possible for small changes.

6. The Sneaky Nature of *e*

All this is very well, but it might surprise you that a table of natural logarithms had been formulated long before *e* was known, and before the modern definition of logarithms even existed.

John Napier (1614) was interested in simplifying computations, and introduced the word logarithm in his *descriptio* (logos- ratio and arithmos- number in Greek). He used a kinematic model of two particles moving along parallel lines- one at constant velocity

(AP), one at decreasing velocity (GP). His painstaking 20-year effort provided the first accurate, comprehensive logarithm table.

The constant *e* appeared in works of Christiaan Huygens, William Oughtred, Nicholas Mercator, and many others over the course of history.

Our discussion of *e*	What it represents
Compound interest	Continuous growth
e^x derivative	Growth proportional to current value
Area under $\frac{1}{x}$ curve	Natural logarithmic scaling
Euler's series	Infinite accumulation

There is still a lot I don't know, more that I don't understand, but you might gather that all these seemingly different definitions are describing the same kind of processes.

Perhaps the most surprising appearance of *e* occurs in probability. Suppose a group of guests checks their hats at a party, and the hats are returned at random. What is the probability that nobody receives their own hat?

Remarkably, as the number of guests grows, this probability approaches $\frac{1}{e}$, which means there is a $\approx 36.78\%$ chance that nobody gets their hat. A constant born in problems of continuous growth appears in questions about random permutations. We would look at the mathematical reasoning, but the word limit begs me to leave this as an exercise to the reader.

7. What can we gather?

Ah, the attempts to decipher secrets.
To look, manipulate, connect, play, derive bliss.
To wrestle with what behaves irrationally.
To contain what transcends our boundaries.

The great physicist Dirac believed rather than chasing accuracy, one should pursue elegance and beauty in equations. When I look at equations like $e^{i\pi} + 1 = 0$, I very much agree.

In sums, in chance, in endless growth
its presence comes to be-
a constant found again and again,
the restless number e.

And what greater joy than to study the universe rigorously, continually, religiously? What greater solace than in the world of perfection and abstraction, in sonnets weaved through symbols and odes addressed to logic?

Not born in one triumphant proof
nor named at first with care
slipping through hidden threads
How many more such things are there?

Yes, I quite agree with Dirac, but not about the fact that poetry and scientific disciplines are incompatible. :)

8. References

- https://mathshistory.st-andrews.ac.uk/HistTopics/e/
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