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<u>From multiplication to vector calculus</u>

<u>a guide on how mathematics gets difficult</u>

Hello dear readers, I do hope you enjoy this essay and enjoy reading it as much as I enjoyed writing it! it is a bit word heavy at the start, but I hope it is a thought provoking read.

1. Introduction

I would like to begin with the origins of mathematics, that being its use to model real-life scenarios. Early civilisations developed numerical systems not out of curiosity, but necessity, in order to keep track of livestock, measure land, and organise trade. I use the word *develop* loosely because these tools weren't a given, nor were the "materials" we used them upon, that being numbers; they were all conceived by a person and then taught and made widespread. I find it an interesting discussion as to whether, if other civilisations exist, they would discover the same mathematical frameworks as us, but whatever your opinion, the order in which mathematical tools were first constructed most definitely fits the purpose of meeting our demands in the real world.

As mathematical thinking developed and more abstract ideas became widespread, certain mathematicians' thought processes began to diverge from the confines of reality. Pure mathematics as a field began to advance for its own sake rather than to solve an underlying problem. In spite of this unique thinking, tools are only tools for as long as they have a use, and there is no better example than the imaginary number system. First conceived—and dismissed—by Cardano in the 16th century, who encountered square roots of negative numbers while solving cubic equations, but without any real-world counterpart to equate these answers to, they were dismissed as fictitious. The very term "imaginary", coined by Descartes, was intended to be derogatory. It took much longer for imaginary numbers to be popularised by mathematicians such as Leonhard Euler, with the famous formula.

And even longer for them to be used to model real-world scenarios such as AC electricity. Now, because of their usage, imaginary numbers are so fundamental they are even taught before university during A-level Further Maths courses.

I mention this because it is around my age that young people such as myself begin to properly consider career options and what they want to do for the rest of their lives, and as prideful as we all are to be mathematicians, a popular choice is one term we cannot use (despite the rather intimidating number of applicants per place for mathematics degrees). I find that one of the key reasons for this is because it's quite intimidating to look ahead at advanced concepts; a single page of an analysis textbook is enough to put many students in shock. It is very easy to get lost and lose the underlying point in pure mathematics, and a practice I have always found helpful is to take a real-world application of a concept, understand it, understand why the maths allows us to solve it, and once I am confident, move on to computations. With this essay, my goal is to work my way up from basic mathematical understandings that everybody—even non-mathematicians—has, and try to build intuition for an advanced concept, hopefully inspiring a few to read more about it for themselves.

2. The building blocks of the universe

To begin, think on a grand scale about the universe: stars burning in outer space, large planets of rock and ice, celestial bodies unfathomably large beyond our comprehension. And yet, one thing can be said about almost every single one of them, and that is that they are always in a state of constant change. Be it the solar system orbiting the black hole at the centre of the Milky Way, or a star fusing hydrogen atoms at its core, or on a smaller scale, burning wooden sticks in a fireplace as you watch the flames dance.

In order for there to be any change in a system, there needs to be two things. The first is energy. Everybody knows you need energy to do things; nothing happens without energy. The question of what energy is is rather abstract, but a good analogy is that energy is like money, and you can't get anywhere without money. If your money is sitting idly in your bank account, nothing changes; a house won't magically appear in your name. Things need to be done with that money, and so that money needs to leave your account. The means of energy transfer in the world is by a force. That's what forces do.

People often think of forces as something that either pushes or pulls on an object, and that's not incorrect, but that's not all they are. Unfortunately, as much as I would love to just transfer money all day, HMRC has other ideas—they are going to tax a small amount when a business makes a transfer. Similarly, when talking about energy transfers, there isn't a single entity to blame for energy loss, but many different mechanisms: heat when your car brakes are applied, or sound when an engine runs.

Just as you need to understand tax to manage finances, when humans build machines, we want to know how much of the force we apply actually goes towards doing useful work. The useful part is what we refer to as the *work done* by a force. Now, how do we measure this? Returning to the push/pull idea, we can measure how far the object has been moved. In other words:

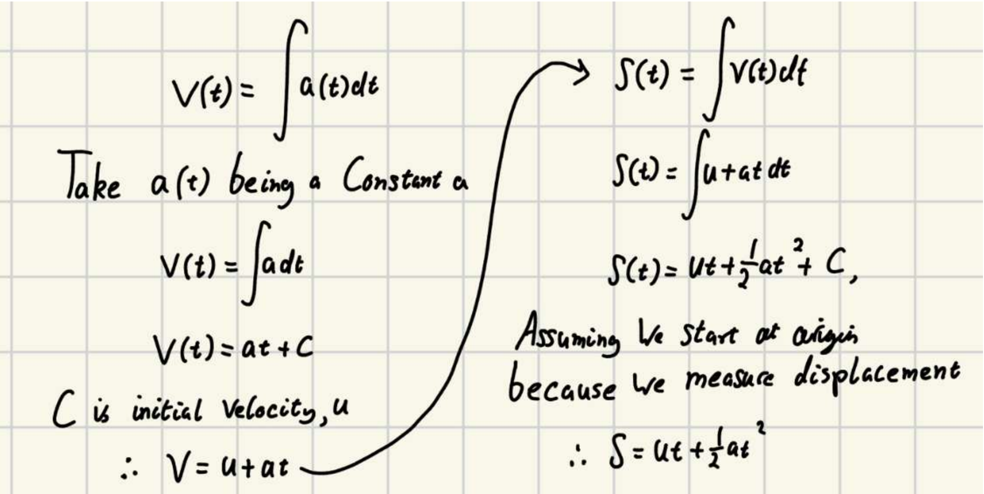
Work done = force × displacement

This will be our main idea for the rest of the essay. It is powerful, but in its current form, it has limitations. So now we begin working our way up the chain.

3. The Physicist's most powerful tool

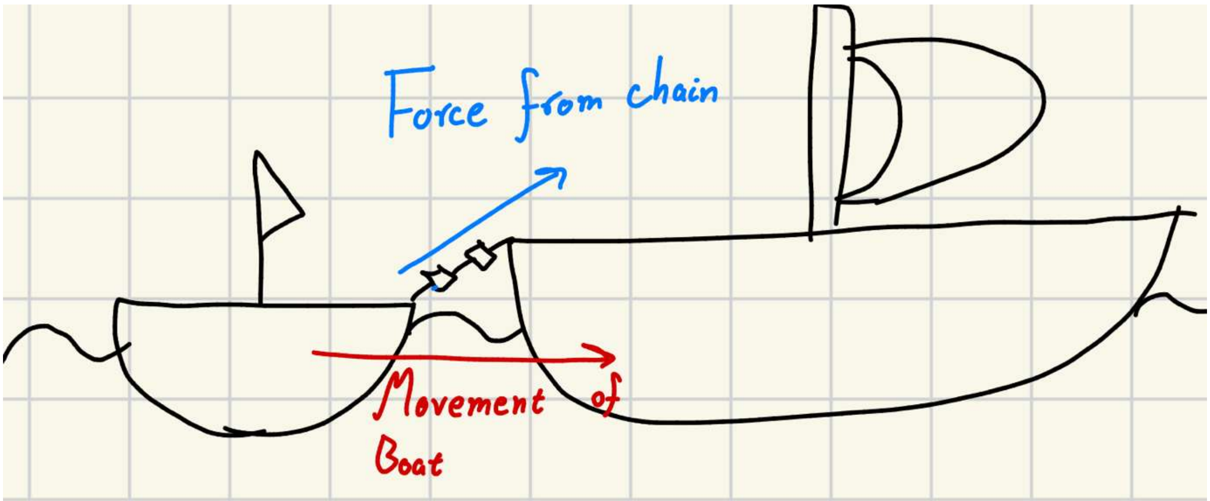
Physicists have many tools at their disposal, and despite using the same underlying principles as mathematicians, their use is often less exact, because real life is not perfect. The most powerful tool in science is the ability to assume. Whether it is assuming an angle is small or that there is no air resistance, assumptions are everywhere, even when you don't expect them.

This can be a good thing, as it simplifies problems. The famous "assume a penguin is a cylinder" joke or "model this cow as a sphere" illustrates how complex systems can be simplified. For example, in kinematics, instead of always integrating acceleration, we consider cases where acceleration is constant and derive the SUVAT equations.



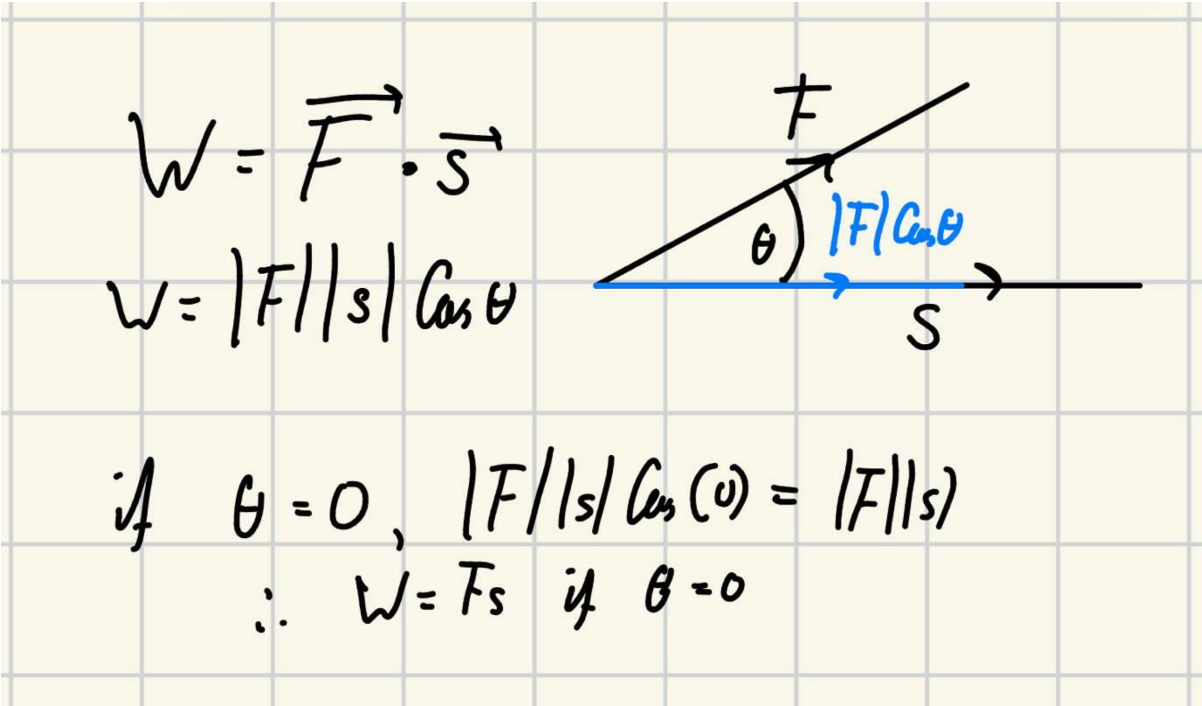
Now most people learn the suvat equations first, and learn how to deal with variable acceleration later and when they see integral definitions it might feel like what they've been taught was a lie or a half truth, taking away from the joy of learning because they concept you spent so long learning about? "actually, that's not the REAL equation, the PROPER WAY is this" or something along those lines is something said all too often but instead of thinking about it as out with the old in with the new, think of it as tweaking the model in your head to encompass more scenarios. This idea of building a model is more gratifying to learn, and the cherry on top is that as shown above using an advanced model if the conditions are right – the problem collapses back into the simpler case you learnt previously.

But back to the star of the show again, W = Fs. What assumptions did we make? Well for starters, we assumed that the direction the force acted in is exactly the same direction the distance acts in. When a boat is towing a smaller vessel upwards, that doesn't cause the boat to start flying diagonally, it causes the boat to move horizontally, so we can say that only a component of the force acts along the line of action (action being the thing that happened – here distance travelled)



Our problem is that the directions don't line up, and in maths whenever we are dealing with things that have both magnitude (how much) AND direction (to where) we use vectors. Opening up our vector tool kit, we can use the scalar product, or dot product of the two forces F . s instead of just Fs. But the intuition behind this is what makes it beautiful.

We know from our trigonometric ratios that cos(θ) = adjacent/hypotenuse. Looking at our vectors, if we take the force to be the hypotenuse and then consider the angle between them θ, then Fcos(θ) gives the component of the force that acts in the same direction as the displacement and then we have the same case we have before. Once again, if the angle between them is 0, then since cos(0)=1, the problem collapses into our earlier formula.



Now it's time we tackle our next assumption. What happens if force isn't constant, like it often isn't. If I'm dragging a heavy box, I'm going to pull then get tired and let go a bit, then once I've gathered the strength give it another TUG, the force won't be constant. In a more realistic situation, Im sure in physics you've seen lots of scenarios where force changes. For example, when you're falling from a plane and air resistance slows you down, most of us have seen that GIF of the force arrows slowly being balanced out as you reach terminal velocity. So how do we measure work done when there isn't one single vector we can multiply by… what do we do when something… is CHANGING… think, ponder it… if only there was a whole field of mathematical operations designed for dealing with changes… some of you might've guessed it – everybody's favourite branch of mathematics, its calculus!

So, let's go back to our equation. We dot the force and displacement vectors. But how about instead of just dotting it once, we split our action into infinite tiny frames. And then in each of those frame dot what value the force has, with the tiny amount that the displacement has increased by, then then add up all those tiny little changes to get the overall work done. Does this process seem familiar? What's precisely what an integral does.

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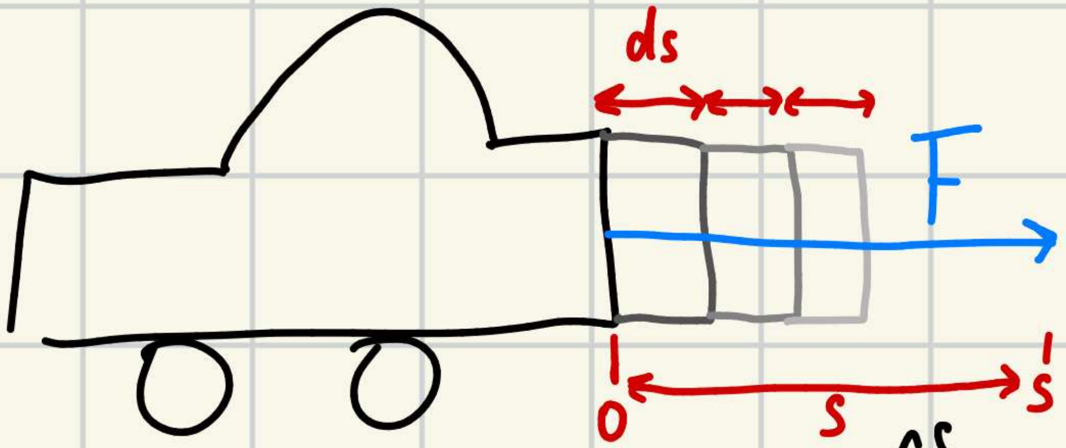
We are multiplying by a constant infinitesimal increase in distance, so we integrate with respect to ds (infinitesimal change in displacement (s)), then the function we are multiplying by is the distance s. If we want to measure the work done over an interval of distance – then we can take our function of force F, and then integrate between initial position (x0) and final position (x) giving us the result of:



For a small interval, $W = \int_0^s F\,ds$

Now we've been changing our force a whole lot, so now let's assume that the path we take isn't linear. We move all over the place. And this is where you really begin to see the vector part of vector calculus. We now treat force and displacement as vectors, with a separate function for the x y and z components.

$$\vec{F} = \begin{pmatrix} F_x \\ F_y \\ F_z \end{pmatrix} \qquad \vec{s} = \begin{pmatrix} s_x \\ s_y \\ s_z \end{pmatrix} \qquad d\vec{s} = \begin{pmatrix} ds_x \\ ds_y \\ ds_z \end{pmatrix}$$

Now recall our integral definition from earlier, we had work done is the integral of force with respect to displacement. But because force and displacement are both vectors, when we integrate, we have to dot them like so.

$$W = \int F\,ds \longrightarrow W = \int \vec{F} \cdot d\vec{s}$$

$$W = \int \begin{pmatrix} F_x \\ F_y \\ F_z \end{pmatrix} \cdot d \begin{pmatrix} S_x \\ S_y \\ S_z \end{pmatrix}$$

$$W = \int F_x\, ds_x + \int F_y\, ds_y + \int F_z\, ds_z$$

often Simplified to

$$W = \int F_x\, dx + \int F_y\, dy + \int F_z\, dz$$

And so that's all I have enough space for in this essay! I would love to have mentioned more about how to deal with parametric forms, answering the question "what if I don't have one as a function of the other, but rather both as functions of time?" but that's a question I encourage you to continue and pursue in your own time. I do hope this essay wasn't a drag and that this topic has captivated you as much as it has me.