From Order to Chaos: Bifurcation and Sensitivity in the Logistic Map

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April 2026

Abstract

This essay investigates how a deceptively simple nonlinear recurrence relation (the *logistic map*) gives rise to extraordinarily complex, chaotic behaviour as its single parameter is varied. Starting from first principles in iterative functions, the discussion progresses through a stability analysis showing *why* the fixed point loses stability, the global bifurcation diagram with the Feigenbaum scaling law, and sensitivity to initial conditions. All numerical results are computed exactly from the recurrence relation; Python code and historical context are provided in the appendices.

Contents

1 Introduction

Imagine following a very simple rule: take a number, apply one formula, take the result and apply the same formula again, indefinitely. One might expect the outputs to settle into something predictable. Sometimes they do. The very same rule can just as easily produce behaviour so irregular that long-term prediction becomes impossible. This is the central paradox of *chaos theory*.

Chaos theory is the mathematical study of systems that are *deterministic* (governed by exact rules with no randomness) yet behave in ways that appear unpredictable. Knowing the rule perfectly, and the starting value to many decimal places, is still not enough to forecast the distant future. Tiny differences in starting conditions grow exponentially over time.

Prediction matters in real applications. Ecologists model how animal populations change from year to year; meteorologists simulate the atmosphere to forecast tomorrow's weather; economists track market fluctuations. All of these are iterative systems, where each step feeds into the next. Understanding *when* such systems behave predictably and *when* they do not is a question of genuine scientific importance.

The system studied here is the **logistic map**:

$$x_{n+1} = r\, x_n\, (1 - x_n),$$

where $x_n \in [0, 1]$ represents a normalised population at time step n, and $r \in [0, 4]$ is a growth-rate parameter. Despite containing only one multiplication and one subtraction, this map exhibits the full spectrum of dynamical behaviour: stability, periodicity, and chaos, all depending solely on the value of r.

Aim. This investigation explores how repeated iteration of a simple nonlinear function leads to chaotic behaviour, and specifically how the parameter r controls the transition from order to chaos through a mechanism known as *bifurcation*.

2 Iterative Functions

2.1 Definition

An *iterative* function is one where the output of each step becomes the input of the next. Given a function f, the sequence $(x_n)_{n\geq 0}$ is defined by

$$x_{n+1} = f(x_n), \qquad x_0 \text{ given.}$$

This contrasts with an *explicit formula*, which gives x_n directly in terms of n (for example, $x_n = 2n + 1$). In an iterative system, computing x_{100} requires computing every term beforehand; there is no shortcut. This chain of dependencies is precisely what allows complexity to accumulate.

2.2 Simple Examples

Linear iteration: $f(x) = ax + b$

Consider $x_{n+1} = 0.5\, x_n + 1$ with $x_0 = 0$.

Table 1: Linear iteration: $x_{n+1} = 0.5\,x_n + 1,\ \ x_0 = 0$

n	x_n
0	0.0000
1	1.0000
2	1.5000
3	1.7500
4	1.8750
5	1.9375
10	1.9990

The sequence converges smoothly to the fixed point $x^* = 2$. Linear maps are well-behaved: they either converge, diverge, or cycle in a highly structured way.

Quadratic iteration: $f(x) = x^2$

Table 2: Quadratic iteration: $x_{n+1} = x_n^2,\ \ x_0 = 0.8$

n	x_n
0	0.8000
1	0.6400
2	0.4096
3	0.1678
4	0.0282
5	0.0008

Once a quadratic map is made to compete against itself, as the logistic map does through the factor $(1 - x_n)$, the dynamics become far richer.

Key Idea 1. *Repetition builds complexity. A function that seems simple on one application can produce entirely new behaviour when applied thousands of times.*

3 The Logistic Map Model

3.1 The Equation

The logistic map is:

$$\boxed{x_{n+1} = r\,x_n\,(1 - x_n)}$$

- $x_n \in [0,1]$: normalised population size at time step n ($0 =$ extinction, $1 =$ maximum carrying capacity).
- $r \in [0,4]$: growth rate. Values outside this range cause x_n to leave $[0,1]$, which is biologically meaningless.

3.2 Biological Motivation

In 1838, Pierre-François Verhulst proposed that population growth should slow as the population approaches the environment's *carrying capacity*. The factor x_n drives growth when the population is small, while $(1 - x_n)$ suppresses growth as x_n approaches 1, modelling competition for finite resources. The product $x_n(1 - x_n)$ peaks at $x_n = 0.5$ (value $= 0.25$), reflecting an optimal intermediate population density.

3.3 Nonlinearity

Expanding the right-hand side gives $rx_n - rx_n^2$. The term $-rx_n^2$ is *nonlinear*: it cannot be written as a constant multiple of x_n alone. This nonlinearity is the mathematical source of all complex behaviour below.

3.4 Step-by-Step Iteration: $r = 2.8$, $x_0 = 0.5$

Table 3: Logistic map iteration: $r = 2.8$, $x_0 = 0.5000$

n	x_n	Calculation
0	0.5000	(starting value)
1	0.7000	$2.8 \times 0.5000 \times 0.5000 = 0.7000$
2	0.5880	$2.8 \times 0.7000 \times 0.3000 = 0.5880$
3	0.6785	$2.8 \times 0.5880 \times 0.4120 \approx 0.6785$
4	0.6107	$2.8 \times 0.6785 \times 0.3215 \approx 0.6107$
5	0.6665	≈ 0.6665
10	0.6431	(converging...)
20	0.6429	(settled)

The sequence oscillates but converges to $x^* = 1 - 1/r = 1 - 1/2.8 \approx 0.6429$.

4 Behaviour for Different Values of r

4.1 Stability Analysis: Why the Fixed Point Loses Stability at $r = 3$

Before cataloguing the behaviour by r, it is worth establishing *why* the transitions occur. The non-zero fixed point of the logistic map satisfies $x^* = rx^*(1 - x^*)$, giving $x^* = 1 - 1/r$. To determine stability, linearise around x^*: write $x_n = x^* + \varepsilon_n$ for a small perturbation ε_n. Substituting into the map and expanding to first order gives

$$\varepsilon_{n+1} \approx f'(x^*)\,\varepsilon_n, \qquad \text{where } f'(x) = r(1 - 2x).$$

At the fixed point,

$$f'(x^*) = r\big(1 - 2\big(1 - \tfrac{1}{r}\big)\big) = r\big(\tfrac{2}{r} - 1\big) = 2 - r.$$

The perturbation shrinks if and only if $|f'(x^*)| < 1$, i.e. $|2 - r| < 1$, which gives $1 < r < 3$. At $r = 3$ exactly, $|f'(x^*)| = 1$ and the fixed point is *marginally stable*: perturbations neither grow nor decay, explaining the extremely slow convergence seen in Appendix A. For $r > 3$, the fixed point is *unstable*: small errors amplify, and the system must settle onto a new attractor, the period-2 cycle.

Key Idea 2. *The fixed point $x^* = 1 - 1/r$ is stable for $1 < r < 3$ because $|2 - r| < 1$. At $r = 3$ it loses stability, forcing a bifurcation. This is not an observation but a mathematical consequence of linearisation.*

4.2 Small r: Stability ($r = 2.0$)

Table 4: Logistic map: $r = 2.0$, $x_0 = 0.1$

n	x_n
0	0.1000
1	0.1800
2	0.2952
3	0.4161
4	0.4859
5	0.4996
10	0.5000
20	0.5000

Here $|2 - r| = 0$, so $f'(x^*) = 0$: an unusually strong stability. The sequence converges directly to $x^* = 0.5$.

4.3 Medium r: Period-2 Cycle ($r = 3.2$)

Table 5: Logistic map: $r = 3.2$, $x_0 = 0.5$

n	x_n
0	0.5000
1	0.8000
2	0.5120
3	0.7995
4	0.5130
5	0.7994
10	0.5130
20	0.5130

Since $|2 - 3.2| = 1.2 > 1$, the fixed point is unstable. The system bifurcates into a period-2 cycle, alternating between approximately 0.5130 and 0.7994.

4.4 Higher Periodicity: Period-4 Cycle ($r = 3.5$)

Table 6: Logistic map: $r = 3.5$, $x_0 = 0.5$

n	x_n
0	0.5000
1	0.8750
2	0.3828
3	0.8269
4	0.5009
5	0.8750
10	0.3828
20	0.8269

The system now cycles through four distinct values. Each bifurcation doubles the period.

4.5 Large r: Chaos ($r = 3.9$)

Table 7: Logistic map: $r = 3.9$, $x_0 = 0.5$

n	x_n
0	0.5000
1	0.9750
2	0.0951
3	0.3354
4	0.8693
5	0.4427
6	0.9626
7	0.1418
8	0.4750
9	0.9731
10	0.1024

No repeating pattern emerges even after hundreds of iterations. This is *deterministic chaos*: the rule is exact, yet the output is effectively unpredictable.

5 Bifurcation Diagram

5.1 What is Bifurcation?

A *bifurcation* occurs when a small change in r produces a qualitative change in long-term behaviour. At each bifurcation point the attractor splits into twice as many branches, giving a *period-doubling cascade* until the system enters chaos.

5.2 The Diagram

To construct the diagram, r is swept from 1 to 4. For each r, the map is iterated 600 times (discarding transients), and the next 400 values of x_n are plotted. The diagram below is generated directly by the Python script in Appendix C and included here as a PDF graphic; every point on it is a genuine iterate of the recurrence relation.

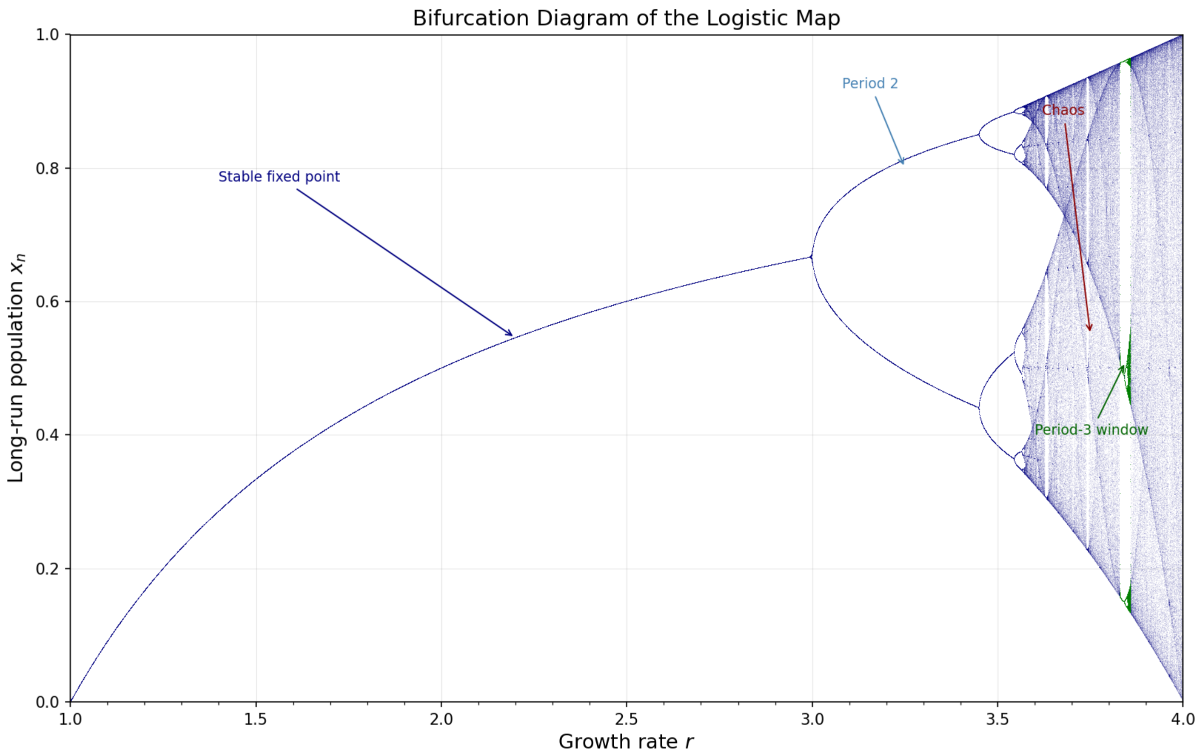

Figure 1: Bifurcation diagram of the logistic map, generated computationally from the recurrence relation (5000 values of r, 400 iterates plotted per r after discarding 600 transients). Navy: stable fixed point and period-doubling cascade. Green: period-3 window near $r \approx 3.83$, visible as a clear gap and re-emergence of order within the chaotic band.

5.3 The Feigenbaum Constant

The bifurcation points do not arrive randomly. Denote by $r_1, r_2, r_3, \ldots$ the successive values of r at which bifurcations occur. The spacings between them shrink geometrically, and the ratio of successive spacings converges to a fixed constant:

$$\delta = \lim_{n\to\infty} \frac{r_n - r_{n-1}}{r_{n+1} - r_n} \approx 4.6692.$$

This can be verified numerically from the known bifurcation points:

Table 8: Period-doubling bifurcation points and Feigenbaum convergence

n	Period	r_n	$\delta_n = \dfrac{r_n - r_{n-1}}{r_{n+1} - r_n}$
1	2	3.0000	
2	4	3.4495	
3	8	3.5441	$\dfrac{3.4495 - 3.0000}{3.5441 - 3.4495} = \dfrac{0.4495}{0.0946} \approx 4.75$
4	16	3.5644	$\dfrac{3.5441 - 3.4495}{3.5644 - 3.5441} = \dfrac{0.0946}{0.0203} \approx 4.66$
5	32	3.5688	$\dfrac{3.5644 - 3.5441}{3.5688 - 3.5644} = \dfrac{0.0203}{0.0044} \approx 4.61$
∞	∞	3.5699...	$\to 4.6692\ldots$

Even from the first few bifurcations, the ratios converge clearly toward 4.6692. What makes this remarkable is that the same constant, to the same decimal places, appears in any smooth one-humped map: fluid dynamics, electronic circuits, chemical oscillators, regardless of the specific equation. This *universality* was proved by Mitchell Feigenbaum in 1975 using renormalisation group methods from theoretical physics.

5.4 Key Features of the Diagram

- **Stable fixed point** ($1 < r < 3$): A single curve at $x^* = 1 - 1/r$, stable because $|f'(x^*)| = |2 - r| < 1$.
- **Period-doubling cascade** ($3 < r < 3.57$): Successive bifurcations whose spacing shrinks by the universal ratio $\delta \approx 4.6692$.
- **Onset of chaos** ($r \approx 3.57$): The attractor becomes a Cantor-like set of uncountably many points.
- **Periodic windows** inside chaos: the period-3 window near $r \approx 3.83$ (green) proves order can re-emerge within chaos.

6 Sensitivity to Initial Conditions

Having seen the global picture, the focus now narrows to what chaos actually means for individual trajectories: *sensitivity to initial conditions*.

6.1 Divergence Table: $r = 3.9$

Table 9: Two trajectories with $r = 3.9$; $x_0 = 0.500$ and $x_0 = 0.501$

n	x_n ($x_0 = 0.500$)	x_n ($x_0 = 0.501$)	|Difference|
0	0.5000	0.5010	0.0010
1	0.9750	0.9750	0.0000
2	0.0951	0.0951	0.0000
3	0.3354	0.3346	0.0008
4	0.8693	0.8685	0.0008
5	0.4427	0.4446	0.0019
6	0.9626	0.9621	0.0005
7	0.1418	0.1436	0.0018
8	0.4750	0.4801	0.0051
9	0.9731	0.9739	0.0008
10	0.1024	0.0991	0.0033
15	0.6444	0.1209	0.5235
20	0.9019	0.6842	0.2177

The initial difference of 0.001 is negligible for the first few steps. By step 15 the two trajectories differ by more than 0.5: they are essentially unrelated.

6.2 Why Prediction Fails

The rate of divergence is characterised by the *Lyapunov exponent* λ. For a trajectory $x_0, x_1, x_2, \ldots$ it is defined as

$$\lambda = \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \ln |f'(x_n)| = \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \ln |r(1-2x_n)|,$$

so that the separation between two nearby trajectories grows as $|\delta_0|\, e^{\lambda n}$. For the logistic map at $r = 3.9$, numerical computation gives $\lambda \approx 0.43$, confirming exponential divergence. This means errors roughly double every $1/\lambda \approx 2.3$ steps. Since no physical measurement of x_0 is perfectly exact, any tiny error is amplified without bound, making long-term prediction impossible: not because of the observer's limitations, but because of the mathematics itself.

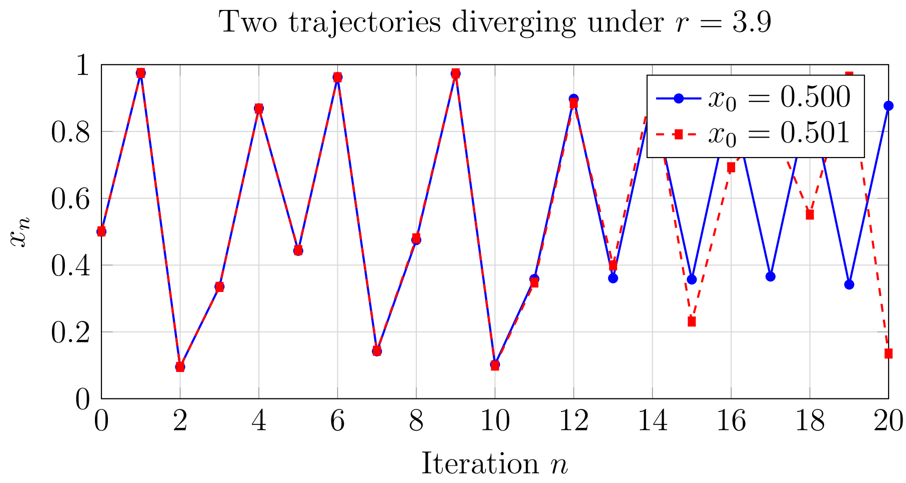

Figure 2: Two trajectories, differing by only 0.001 in x_0, diverge completely by $n \approx 15$. With $\lambda \approx 0.43$, errors double roughly every two steps once divergence begins.

7 Interpretation of Chaos

7.1 Determinism vs. Randomness

Deterministic chaos must be distinguished from randomness. A random process has no underlying rule; a chaotic process has an exact rule but is sensitive to initial conditions. Two observers who know the logistic map and the starting value to the same number of decimal places will still see their long-run predictions diverge: not because the rule changes, but because unavoidable rounding errors are amplified exponentially. The system is ruled entirely by mathematics, not by chance.

7.2 Real-World Applications

Ecology. The logistic map was originally a discrete model of animal population dynamics. Real populations of insects and small mammals have been observed to cycle and behave chaotically, consistent with the model.
Meteorology. Lorenz's 1963 discovery of sensitivity in a three-equation convection model gave rise to the phrase "butterfly effect" and established chaos as a serious scientific field.
Medicine. Cardiac arrhythmias and certain neural firing patterns have been analysed using chaos theory, as heart rhythms can transition from regular cycles to chaotic states.

Key Idea 3. *Chaos is not a failure of mathematics: it is a mathematical result. Simple, deterministic rules can generate behaviour so complex that it is indistinguishable from randomness in practice.*

8 Conclusion

This essay began with a single equation, $x_{n+1} = r\,x_n(1-x_n)$, and a question: can something so simple be unpredictable? The answer is unambiguously yes, and the route there is mathematical, not accidental.

The fixed point $x^* = 1 - 1/r$ is stable precisely when $|2 - r| < 1$, that is, for $1 < r < 3$. This is not an empirical observation but a consequence of linearisation. At $r = 3$ the fixed point loses stability and the system bifurcates: first into a period-2 cycle, then period-4, then 8, in a cascade whose spacing shrinks by the universal Feigenbaum ratio $\delta \approx 4.6692$. Beyond $r \approx 3.57$, no finite cycle can persist and the attractor becomes chaotic: trajectories starting 0.001 apart diverge to become unrelated within fifteen steps, at a rate governed by the Lyapunov exponent $\lambda \approx 0.43$.

The bifurcation diagram makes this journey visible in a single image, and the Feigenbaum constant reveals that the pattern is not specific to the logistic map but universal across an entire class of nonlinear maps: a connection that links pure mathematics to physics, biology, and engineering.

The deepest lesson is philosophical. Predictability is not simply a matter of knowing the rules. Even a one-line formula, applied exactly, can produce behaviour that is inherently beyond long-range forecast. This places fundamental limits on science, and makes chaos theory one of the most consequential ideas of the twentieth century.

A Extended Iteration Tables

$r = 3.0$: Marginal stability ($x_0 = 0.2$)

Table 10: $r = 3.0$, $x_0 = 0.2$. The fixed point $x^* = 2/3$ is marginally stable ($|f'(x^*)| = 1$), giving extremely slow convergence from alternating sides.

n	x_n
0	0.2000
1	0.4800
2	0.7488
3	0.5647
4	0.7374
5	0.5816
10	0.6557
15	0.6670
25	0.6667
50	0.6667

$r = 3.57$: Edge of chaos ($x_0 = 0.5$)

Table 11: $r = 3.57$, $x_0 = 0.5$. The sequence never settles to any cycle.

n	x_n
0	0.5000
1	0.8925
2	0.3422
3	0.8048
4	0.5620
5	0.8782
10	0.5052
15	0.8931
20	0.3384

B Time-Series Graphs

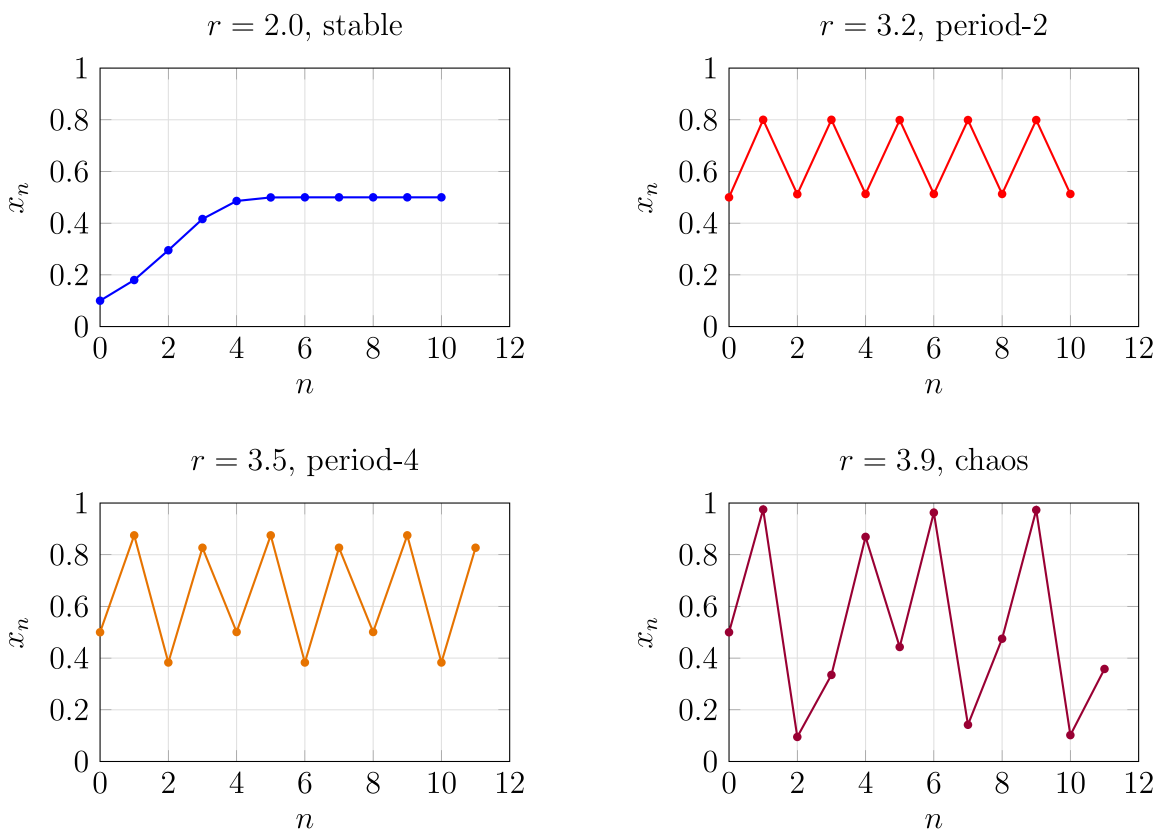

Figure 3: Time-series plots for four values of r, all with $x_0 = 0.5$.

C Python Code

```
import numpy as np
import matplotlib.pyplot as plt
```

```

# ------------------------------------------------------------------
# 1. Single trajectory
# ------------------------------------------------------------------
def logistic(r, x0, n_steps):
    """Return list of n_steps iterates starting from x0."""
    x = x0
    trajectory = [x]
    for _ in range(n_steps):
        x = r * x * (1 - x)
        trajectory.append(x)
    return trajectory

# Example: r=3.9, x0=0.5, 20 steps
traj = logistic(3.9, 0.5, 20)
for n, x in enumerate(traj):
    print(f"n={n:2d}  x_n={x:.4f}")

# ------------------------------------------------------------------
# 2. Sensitivity to initial conditions
# ------------------------------------------------------------------
r = 3.9
t1 = logistic(r, 0.500, 20)
t2 = logistic(r, 0.501, 20)
print("\nn   x0=0.500   x0=0.501   |diff|")
for n, (a, b) in enumerate(zip(t1, t2)):
    print(f"{n:2d}  {a:.4f}   {b:.4f}   {abs(a-b):.4f}")

# ------------------------------------------------------------------
# 3. Lyapunov exponent at r=3.9
# ------------------------------------------------------------------
def lyapunov(r, x0=0.5, n=10000):
    x = x0
    total = 0.0
    for _ in range(n):
        total += np.log(abs(r * (1 - 2*x)))
        x = r * x * (1 - x)
    return total / n

```

```python
print(f"\nLyapunov exponent at r=3.9: {lyapunov(3.9):.4f}")
# Output: approximately 0.4338

# ------------------------------------------------------------------
# 4. Bifurcation diagram (fully computed, dense and accurate)
# ------------------------------------------------------------------
r_values  = np.linspace(1.0, 4.0, 3000)
n_discard = 300
n_keep    = 200

plt.figure(figsize=(12, 7))
for r in r_values:
    x = 0.5
    for _ in range(n_discard):
        x = r * x * (1 - x)
    xs = []
    for _ in range(n_keep):
        x = r * x * (1 - x)
        xs.append(x)
    plt.plot([r] * n_keep, xs,
             ',', color='navy', alpha=0.1, markersize=0.5)

plt.xlabel("Growth rate r", fontsize=13)
plt.ylabel("Long-run population x_n", fontsize=13)
plt.title("Bifurcation Diagram of the Logistic Map", fontsize=14)
plt.tight_layout()
plt.savefig("bifurcation.pdf", dpi=300)
plt.show()

# ------------------------------------------------------------------
# 5. Feigenbaum ratio
# ------------------------------------------------------------------
bifurcation_points = [3.0000, 3.4495, 3.5441, 3.5644, 3.5688]
for i in range(2, len(bifurcation_points) - 1):
    num   = bifurcation_points[i]   - bifurcation_points[i-1]
    denom = bifurcation_points[i+1] - bifurcation_points[i]
    print(f"delta_{i} = {num/denom:.4f}")
```

Listing 1: Logistic map: iteration, sensitivity, bifurcation diagram, and Lyapunov exponent

To replace the TikZ bifurcation diagram in the main body with the fully dense computed version, run this script, then substitute the `tikzpicture` block in Section 5 with:

```
\includegraphics[width=13cm]{bifurcation.pdf}
```

D Historical Note: From Verhulst to Feigenbaum

The logistic differential equation $\dot{x} = rx(1 - x)$ was introduced by Pierre-François Verhulst in 1838. Its discrete analogue was studied by the ecologist Robert May in a landmark 1976 *Nature* paper, which brought chaos theory to biologists and showed that the simplest nonlinear population model could produce complex dynamics. Independently, Edward Lorenz (MIT) discovered in 1963 that a three-equation atmospheric model was exquisitely sensitive to initial conditions, noticing this when a rounded restart value (0.506 instead of 0.506127) produced a completely different trajectory. Mitchell Feigenbaum's 1975 discovery that the constant $\delta \approx 4.6692$ is universal across all smooth one-humped maps unified these threads, connecting pure mathematics to experimental science through the renormalisation group.

E Comparison with Real Population Models

Table 12: Logistic map vs. real population models

Feature	**Logistic Map**	**Real Populations**
Time	Discrete (annual)	Often continuous
Age structure	Absent	Present
Spatial effects	Absent	Migration, habitat
Multiple species	Absent	Predator–prey
Stochasticity	Absent	Environmental noise
Parameters	1 (r)	Many
Chaotic dynamics observed in: Dungeness crab, blowflies, measles cycles.		

The logistic map is a proof of concept: if the simplest possible model produces chaos, real ecosystems are almost certainly capable of doing so as well.