Functional Equations and Various Means

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1. Introduction

The relationships in this essay almost came to me in a dream, Ramanujan style. It was literally a random Tuesday night at 1AM, and the thought was so interesting I had to write it down. It still ended up keeping me up all night, so I hope this was worth it. Mathematical discoveries are really just happy little accidents!

It all started with the arithmetic mean. We all love the arithmetic mean: it was the first taste of the mathematical process, one of the first computations we learned how to do after learning the basic operations. In school, you were likely told that the arithmetic mean is the "average" of a bunch of numbers, or elements. Later, you were then told how to compute it with a generalized formula:

$$\frac{x_1 + x_2 + \ldots + x_n}{n}.$$

In this formula, you add all the elements together, and then divide by the number of elements that you have. Using the arithmetic mean as an operation with an arbitrary amount of elements gets really complicated, so for now I'll be using the average of only two numbers instead, which is this formula:

$$\frac{x_1 + x_2}{2}.$$

The average is one of the most practical, useful functions[1] in all of math. So how does this function relate to high-level, abstract, pure mathematics? What exotic properties can come from analyzing such a seemingly simple function?

2. A Little Algebra

To get started, we need to rewrite the average with a little bit of algebra. Is there another way to express the function that could prove useful? What if we tried to separate x_1 and x_2?

Firstly, let's rewrite the equation for our specific form with two numbers. x_1 is now a, and x_2 is now b. Set the whole thing equal to an arbitrary value. Let's say x. The resulting expression should look familiar:

$$x = \frac{a+b}{2}$$

Multiply both sides by 2.

$$2x = a + b$$

Now, subtract a or b to bring it to the other side. Let's say b.

$$2x - b = a$$

Lastly, a crucial step. We need to make the equation have the same form on both sides. We can do this by subtracting one x to the other side. With a little rearranging we get:

$$a - x = x - b.$$

This is a very important equation due to the fact that it uses the same operation on both sides of the equal sign. Note that "x" refers to the arithmetic mean (in this case), and a and b are arbitrary constants.

3. Getting Geometric

In geometry class, we unlocked a new type of mean. Instead of finding the "middle" of two numbers using addition, we find it using multiplication. The geometric mean is also very useful, especially so in cases with exponential growth. The general formula is similar to the arithmetic mean, but addition is replaced with multiplication and division is replaced with a root. It looks like this:

$$x = \sqrt[n]{x_1 x_2 \dots x_n}$$

And the simplified formula (for 2 constants) looks like this:

$$x = \sqrt{ab}$$

What if we could apply the same process we used on the arithmetic mean? First, we have to separate the constants. Square both sides of the equation.

$$x^2 = ab$$

Next, divide both sides by *a* or *b*. Once again, we will use *b*.

$$\frac{x^2}{b} = a$$

Lastly, to give both sides the same form, divide both sides by *x*. This process should feel familiar by now. Doing a little rearranging we get:

$$\frac{a}{x} = \frac{x}{b}$$

As a reminder, here is the equation we obtained from the arithmetic mean:

$$a - x = x - b$$

Notice any similarities? In both equations, we have *a* and *x* on one side, and on the other we have *x* and *b*. Both equations also have only one operation applied to both sides. Do you know what that means?
We can generalize!

4. The Arrival of Functions

We can express the subtraction and multiplication operations with a function *f* that takes in TWO variables. As an example, if *f* represented subtraction, then the function *f(m, n)* would equal *m-n*. For the above equations, we can substitute the operation with *f*, creating this functional equation *f(a, x) = f(x, b)*.

In the case of the arithmetic mean, *f* would represent subtraction. In the case of the geometric mean, *f* would be division. Because *f* can represent any arbitrary operation, let's have some fun and try to generate new types of means. What if *f* represented addition? Then we get the equation:

$$a + x = x + b$$.

That's not good. *x* only has a valid solution when *a=b*. How about multiplication?

$$ax = xb$$

Once again, *x* only has a valid solution when *a=b*.

It seems like the functional equation only yields meaningful results when *f* represents an operation that does not have the commutative property. Otherwise, you could get rid of x by taking the inverse of the operation in the function.

In the case of *f(m, n) = m+n*, the inverse operation would be subtraction. Taking the inverse operation always yields the equation *a=b*. Fortunately, addition and multiplication are the only relevant operations that have the commutative property.[2]

Let's try another operation. Exponentiation should work, as it is not commutative. Applying the function yields

$$a^x = x^b$$.

Now, we can solve for *x*. The process of doing so is possible but is very difficult and does not need to be thoroughly explained in this exploration. "The proof is left as an exercise to the reader"

$x = e^{-W(-\frac{\ln a}{b})}$, where W is the Lambert W function[3]

It seems like even rather simple definitions of *f* can give very complicated results. Although this expression arises naturally from the functional equation, it does not satisfy the properties we typically expect from a mean. This is because swapping the values of *a* and *b* in the above expression changes the result, but it doesn't for the means. This property is called symmetry.

We've explored a lot of ways *f* can generate "invalid means". So, how can we guarantee that *f(m, n)* produces a valid, symmetric mean?

5. The Mother of All Means

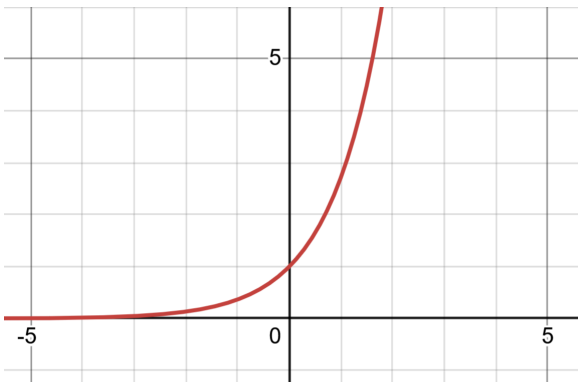
Here, we need to be a bit more mathematically precise, so we can create a formula that has no inconsistencies.

Firstly, we want an expression that produces a value x that is symmetric in a and b, and that x is internal (that it actually lies between a and b).

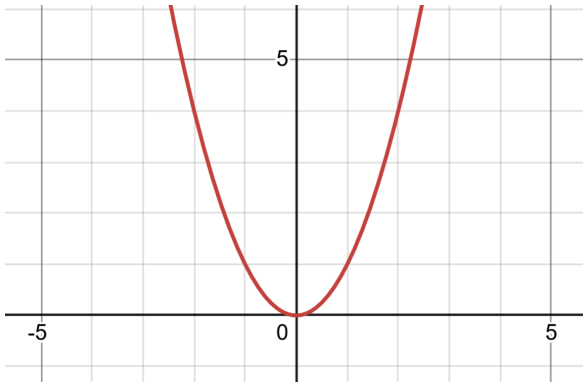
To do this, we introduce a transformation g that maps our original numbers from a set D_1 to a new set D_2. The transformation must be strictly monotone, meaning it preserves order: larger numbers in D_1 remain larger (or consistently smaller, if decreasing) in D_2. This ensures that the process treats a and b fairly and that the resulting mean stays internal.

We also require g to be invertible, so that after finding a midpoint in D_2, we can return to a meaningful value in D_1. Finally, g should be continuous, so that small changes in the inputs produce small changes in the result. As an example, here is one function that satisfies our criteria, and one that doesn't:

$g(x) = e^x$



$g(x) = x^2$



The second function doesn't meet our criteria as it is not strictly monotone (the function changes from decreasing to increasing at $x=0$).

Next, to measure "distance", the function should depend on two inputs, change sign when the inputs are swapped, and become zero when the two inputs are equal. The simplest expression that satisfies these properties is

$$g(m) - g(n).$$

We can now redefine *f(m, n)* as

$$f(m, n) = g(m) - g(n).$$

Now, we plug this into our original expression *f(a, x) = f(x, b)* to get

$$g(a) - g(x) = g(x) - g(b).$$

Doing some algebra, we get to this expression

$$g(x) = \frac{g(a) + g(b)}{2}.$$

Now, all that's left to do is to apply g^{-1} so we can solve for *x*.

$$x = g^{-1}(\frac{g(a) + g(b)}{2})$$

But wait, this formula is really easy to generalize! Notice how it sort of looks like the formula for the arithmetic mean. This means we can "backtrack" from the two number formula to get to the generalized formula:

$$x = g^{-1}(\frac{g(x_1) + g(x_2) + \ldots + g(x_n)}{n}).$$

And this formula actually has a name! It is called the "quasi-arithmetic mean". It is actually most commonly expressed using summation notation, like this:

$$M_g(n_1, n_2 \ldots n_n) = g^{-1}(\frac{1}{n}\sum_{k=1}^{n} g(n_k))$$

This formula looks a lot more complicated, but it's really just the same thing: you sum the values $g(n_1)$ (or, in our equation, $g(x_1)$) to $g(n_k)$, divide by *n*, and then apply g^{-1}. It's the exact same process we used to arrive at the formula. The left side is just how you write "the mean of these numbers". However, for the rest of this exploration, we will use the formula we derived as it is easier to compute with.

6. Implications and Special Cases

The quasi-arithmetic mean is a general way to define a "mean" that unifies many classical means (arithmetic, geometric, harmonic, etc) under a single formula. Those classical means are actually just specific cases of this general formula! It allows you to find a "middle point" of a set of numbers even when the natural notion of "distance" or averaging isn't just ordinary addition. It also has various practical applications: it serves as the foundation for many concepts in probability, finance, and data science.

As an example, let's create the harmonic mean formula for two numbers, one we haven't seen yet, using the quasi-arithmetic mean. First, let *g(x) = 1/x*, and plug it into the two-variable formula:

$$x = g^{-1}\left(\frac{g(a) + g(b)}{2}\right) = g^{-1}\left(\frac{\frac{1}{a} + \frac{1}{b}}{2}\right)$$

Now, since *g(x) = 1/x*, $g^{-1}(y) = 1/y$.[5] Plug in the expression above for *y*, and we get

$$x = \frac{2}{\frac{1}{a} + \frac{1}{b}},$$

which is exactly the harmonic mean!

7. Conclusion

In this exploration, we saw how a simple concept, the arithmetic mean, can lead to a rich world of mathematical ideas. By generalizing the process of "averaging" through functional equations and transformations, we were able to derive the quasi-arithmetic mean, a seemingly complicated equation that encompasses the conventional means as special cases. This not only reveals the deep structure behind a simple formula, but also shows how choosing different transformations allows us to define new, meaningful ways of finding averages for a set of numbers. It's strange to think that something as ordinary as an average can open the door to abstract mathematics. Sometimes, exploring a simple idea just a little further than expected is enough to reveal something unexpectedly beautiful.

Footnotes

1. Technically it's not really a function in this state. I will make many more "incorrect" vocabulary choices in this essay, some I'm aware of and some I'm not. It's for the sake of the non-mathematicians. And my mental health.

2. There are other commutative operations that exist. Addition and multiplication are the only operations that are useful or applicable in this scenario.

3. The Lambert W Function is defined as the inverse of the function $f(W) = We^W$.

4. We assume the numbers lie in an interval of the real numbers, and that g is defined and strictly monotone on that interval.

5. This is true as 1/x is its own inverse. ($y=1/x \rightarrow x=1/y$, $xy=1$, $y=1/x$)

References

https://en.wikipedia.org/wiki/Quasi-arithmetic_mean
https://en.wikipedia.org/wiki/Harmonic_mean
https://cs.uwaterloo.ca/research/tr/1993/03/W.pdf
https://arxiv.org/pdf/2003.01973