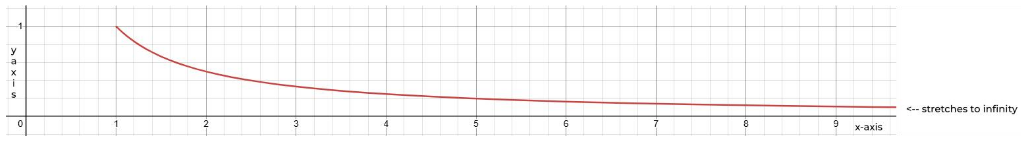
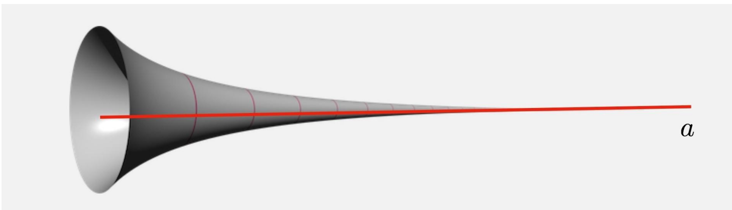
Gabriel's horn / Painter's paradox:

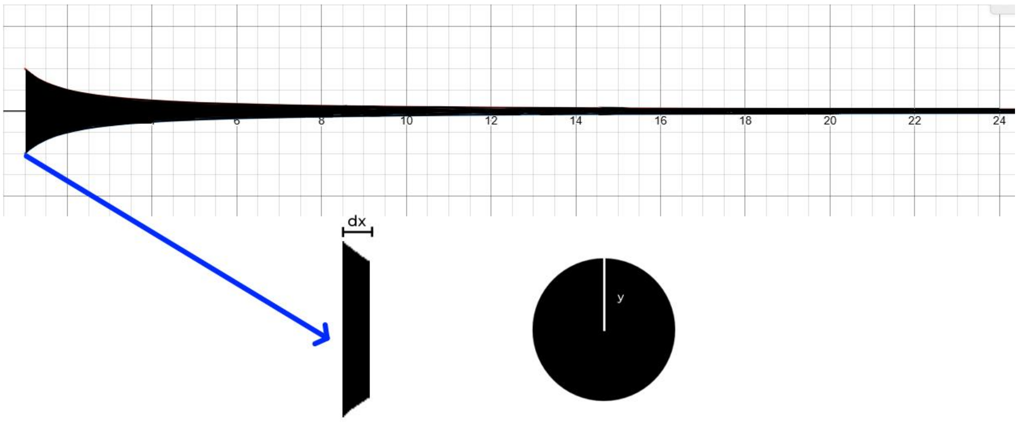
Imagine you have a horn that is stretched to infinity. What do you suppose will be its surface area, and volume? Infinite… right?
Wrong.
This is the painter's paradox. The Gabriel's horn is constructed by plotting the function $y = \frac{1}{x}$ in the interval [1,∞].



This is then rotated around the x-axis to get:



The Gabriel's horn actually has a finite volume, and an *infinite surface area.* An idea of what this would mean is if you try to fill the inside of the horn with a liquid (like water, or paint), it will eventually fill it up. But if you tried to cover the surface with paint, you will never be able to do so - you would require an infinite amount of paint to cover it all up.
So… how?



If we divide the horn into tiny segments / cylinders, such that each has a width of dx, and a radius of y, we can find the volume to be the integration of the area of each cylinder.

$$V = \int_{b}^{a} \pi y^2 dx$$

We set the limits of x to be 1 to infinity, since that is the interval of x, and we plug in the value of $y = \frac{1}{x}$.

$$V = \int_{1}^{\infty} \pi (\frac{1}{x})^2 dx$$

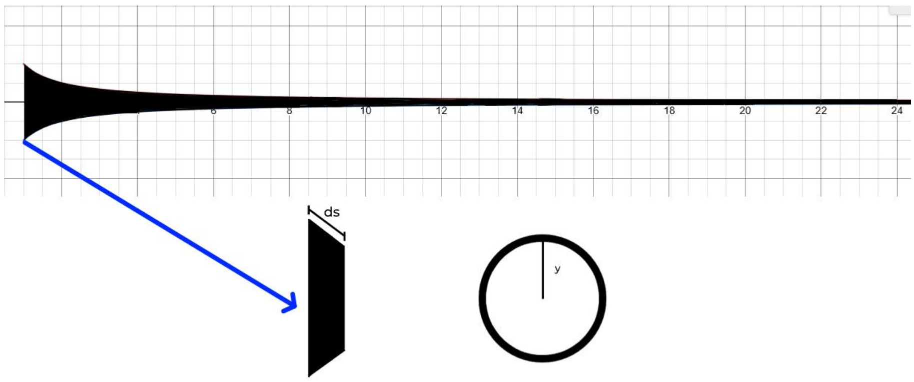
$$V = \pi [-\frac{1}{x}]_{1}^{\infty}$$

$$V = \pi [0 - (-1)]$$

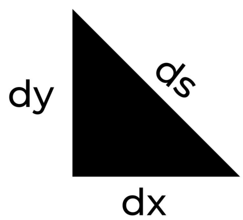
$$V = \pi$$

What this means is that to fill the horn completely, we will need π volume of paint.
We have proved that the volume of this horn is not infinite. What about the surface area? Here we divide the horn into hollow segments / circles, where y is the radius of each circle, and ds is the slope. We cannot use calculate the integration of ds, as we do not know the intervals of s. Also,

we can not ignore the slope here as doing so will give us a greater error than it would have in case of the volume.



Notice how by making components of ds (dx, and dy), we get a right angle triangle?



Using pythagorean theorem, we can find the relation between ds, and dx and dy:

$$ds^2 = dx^2 + dy^2$$

$$ds = \sqrt{dx^2 + dy^2}$$

Taking dx^2 common will give us:

$$ds = \sqrt{1 + (\frac{dy}{dx})^2}\ dx$$

Here, $\frac{dy}{dx} =- \frac{1}{x^2}$:

$$ds = \sqrt{1 + \frac{1}{x^4}}\ dx$$

We can find the surface area by taking integration of the product of the circumference of each circle, and the radical length, ds.

$$A = \int_b^a 2\pi r * \sqrt{1 + (\frac{dy}{dx})^2}\ dx$$

$$A = \int_1^\infty 2\pi(\frac{1}{x})\sqrt{1 + \frac{1}{x^4}}\ dx$$

Here we can use substitution, and then integration by parts, but an easier way to solve this would be to focus on $\sqrt{1 + \frac{1}{x^4}}$. For whatever value of x, the minimum value of this part will be 1. $\frac{1}{x^4}$ can not have a value less than 0. For 0, the answer will be 1, and for any value greater than 0, the value will be greater than 1. So:

$$A \geq 2\pi \int_1^\infty \frac{1}{x} dx$$

$$A \geq 2\pi[ln(x)]_1^\infty$$

The natural log of infinity is equal to infinity, and anything multiplied to, or subtracted from infinity is always going to be infinity. So we can take the whole term as infinity:

$$A \geq \infty$$

Anything greater than infinity will be infinity, so we can set the surface area, A equal to infinity:

$$A = \infty$$

We have now proven that a shape with infinite surface area, and finite volume does indeed exist.

What does this even mean?

A way to understand is if you take a clay of some volume, and roll it into a snake. If you roll the clay to make it as twice as long as before, you will have a new surface area that is half the original area, due to the reduced radius - which is caused since the volume needs to be constant. If you keep rolling this snake to make it infinitely long, the volume will remain equal to the original volume, but the surface area will be infinite.

References:

https://blog.plover.com/math/gabriels-horn.html
https://www.youtube.com/watch?v=yZOi9HH5ueU
https://math.jhu.edu/~brown/courses/s12/ExtraProblems/Section7.8Problem.pdf