Generalisation of Pascal's Triangle to Higher Dimensions

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Contents

1 Introduction

In this paper, the idea of Pascal's triangle is extended to higher dimensions. In doing so, the combination function (nCr) is also extended to any number of variables which leads to a fast method for calculating coefficients of *n-nomial* expansions.

2 A recap of Pascal's triangle

Attached below is a picture of the first eight rows of Pascal's triangle.[1]

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1

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[1] Wikipedia, *Pascal's Triangle*, accessed 11 April 2026

In this paper, the solitary *1* at the vertex of the triangle is taken to be Row 0; *1 1* to be Row 1, and so on. Within the nth row of the triangle, there are $n+1$ elements: the leftmost *1* is taken to be the 0^{th} element, and the rightmost *1* to be the nth element. Then, for the r^{th} element in the n^{th} row of the triangle, its value is given by:

$$\binom{n}{r} = \frac{n!}{r!\,(n-r)!} \tag{1}$$

which is the number of ways of choosing r elements from a collection of n. Initially this connection with combinatorics seems mysterious, but a shift in perspective can show why the connection exists. This shift is to consider each number in the triangle as the number of paths from the vertex that you can take to reach said number. At each stage along a path, you can move down to the left or to the right, so all paths to reach the r^{th} element of the n^{th} row are those where you move to the right r times. For example, to reach the 1st element of the 3rd row, your path can be LLR, LRL, or RLL, which is 3 paths, equal to the value of $\binom{3}{1}$. Thus, the selection of r right turns from n possible turns is where the $\binom{n}{r}$ value comes from.

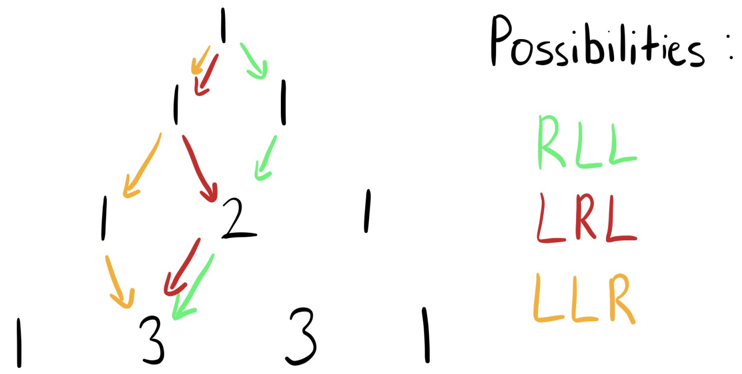

Figure 1: The ways of reaching the leftmost 3

Because of this link, Pascal's triangle can also be used to find the coefficients of terms in a binomial expansion, that is, the expansion of an expression in the form $(x+y)^n$, where $n \geq 0$. Specifically, the numbers in the nth row of Pascal's triangle give the coefficient of the terms in the expansion of $(x+y)^n$, in descending powers of x. The reason this works is because it is fundamentally the same process of choosing to go either left or right down the triangle. If you consider this intermediate expansion of $(x+y)^6$:

$$(x+y)^6 = (x+y)(x+y)(x+y)(x+y)(x+y)(x+y) \tag{2}$$

The coefficient of the x^4y^2 term is the number of ways of going through all 6 brackets and picking x 4 times, which is the same as choosing to go right 4 times as you make your way down the triangle to the 6th row. For example, if the 4th row of Pascal's Triangle is considered, it can be deduced that the expansion of $(x+y)^4$ is:

$$x^4 \, + \, 4x^3y \, + \, 6x^2y^2 \, + \, 4xy^3 \, + \, y^4 \tag{3}$$

Of course, this shortcut is only useful if a method is known for calculating the rows of Pascal's triangle. One way to think about generating the $(n+1)^{\text{th}}$ row, given the n^{th} row, is that each element in the nth row "passes down" a copy of itself diagonally to the right and the left, and then the copies that "overlap" are added together. This is illustrated below, showing how the third row can be used to generate the fourth:

Row 3:		1		3		3		1	
Passing down:	1		1+3		3+3		3+1		1
Row 4:	1		4		6		4		1

3 Into the third dimension: Pascal's pyramid

With this interpretation of generating the triangle, the idea can be extended to the next dimension and Pascal's tetrahedron (or Pascal's Pyramid if you want a catchier name) can be constructed. As with Pascal's Triangle, it starts with a 1 at the top, and that 1 passes down a copy of itself to the three imaginary vertices of a tetrahedron beneath it, forming a new layer. Then this process is repeated with our new layer, and so on and so forth to form subsequent layers. Below is a top-down view of the first few layers:

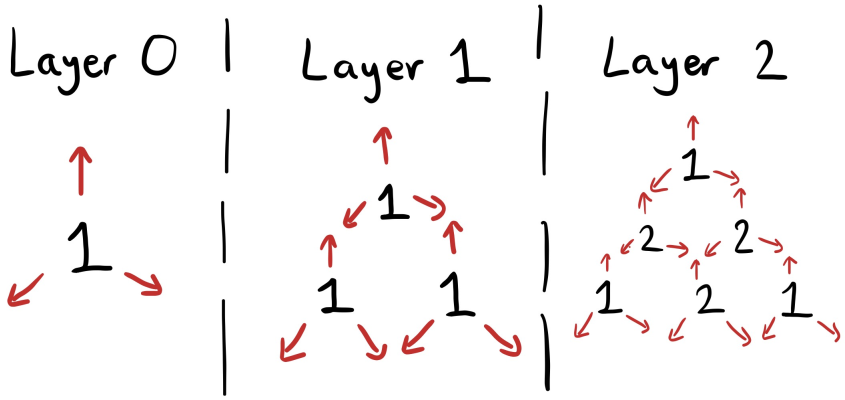

Figure 2: Buidling up layers of Pascal's Pyramid

Given these layers, it is natural to ask if there is a formula to generate the numbers within them, as there is for Pascal's triangle, and whether these numbers have any links to expansions or choosing objects. Rather curiously, there *is* a link: the numbers in each layer of Pascal's tetrahedron correspond to the coefficients of the terms in the trinomial expansion. Here on the left we have the third layer of the pyramid, where n = 3, which has the coefficients of the expansion of $(x+y+z)^3$, and on the right are the corresponding terms that are attached to each of these coefficients.

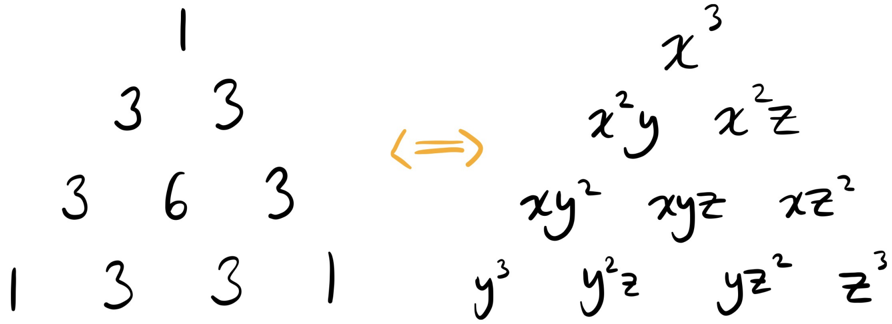

Figure 3: The coefficients in layer 3, and their respective terms

Using this layer of the pyramid, it can be deduced (without actually expanding a single bracket) that

$$(x+y+z)^3 = x^3+3x^2y+3x^2z+3xy^2+6xyz+3xz^2+y^3+3y^2z+3yz^2+z^3 \quad (4)$$

In Pascal's triangle, as you move right in a row of the triangle the exponent of the y term increases and the exponent of the x term decreases. As you move left, the opposite occurs. Within each layer of Pascal's tetrahedron, as you move away from the top vertex of the triangle, the x exponent decreases; as you move away from the bottom left vertex the y exponent decreases; as you move away from the bottom right vertex the z exponent decreases. This can be most clearly seen with the x terms given the orientation in which the layer has been drawn.

This pattern holds for every layer, and this concrete algebraic interpretation can be used to derive the formula for all numbers in the pyramid.

Consider any generic term in the expansion of $(x+y+z)^n$, which will have the form $(x^r)\,(y^s)\,(z^{n-(r+s)})$. It is again useful to write out the expansion as the product of n brackets.

$$(x+y+z)^n = \underbrace{(x+y+z)(x+y+z)...(x+y+z)}_{n\,times} \tag{5}$$

To obtain the desired term, it must be considered that of our n brackets, it is necessary need to choose r number of x's, and then from the remaining $n-r$ brackets, it is necessary to choose s number of y's. Hence, the value to be calculated is

$$\binom{n}{r} \cdot \binom{[n-r]}{s} = \frac{n!}{r!\,(n-r)!} \cdot \frac{(n-r)!}{s!\,(n-r-s)!} = \frac{n!}{r!\,s!\,(n-r-s)!} \tag{6}$$

This is how the combination function can be extended to 3 variables, which will be notated $\binom{n}{r,\,s}$, and gives a much quicker way to calculate the coefficients of a trinomial expansion.

4 Generalising to all dimensions

Now for the fun part, where this is extended to any number of variables. From our two examples, Pascal's triangle and Pascal's tetrahedron, it is clear that the simplest possible shape in n dimensions, combined with the idea of 1 passing down copies of itself will generate a shape consisting of layers a dimension lower (lines for Pascal's triangle, triangles for Pascal's tetrahedron) that encode the coefficients of the terms of an *n-nominal* expansion. This can also be shown to be true for the 1D case, where Pascal's line arises - a vertical line of ones - representing the coefficients of the monomial expansion, which are always 1.

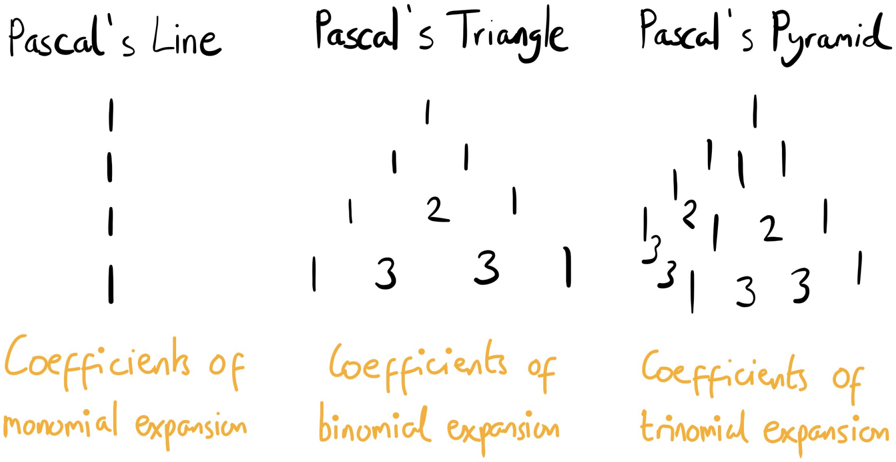

Figure 4: The first 3 terms of the Pascal shape family

The reason why this will work in all dimensions is that the $n+1$ vertices of an n-dimensional shape give a physical representation of picking one of n potential terms: for example, moving from one of the 3 vertices of a triangle to another gives 2 choices, which corresponds to choosing either x or y from one of the brackets. Similarly, moving from one of the 9 vertices of an 8D tetrahedron (called an 8-simplex) to another gives 8 choices, which corresponds to choosing one of the eight variables that would be present in a octanomial expansion.

Even though visualising even the simplest 4D shape (the pentachoron) is impossible, visualising its layers (which themselves are tetrahedrons) is feasible. The first 3, shown below, correspond to the coefficients of the terms in $(w+x+y+z)^0$, $(w+x+y+z)^1$and $(x+w+y+z)^2$ respectively.

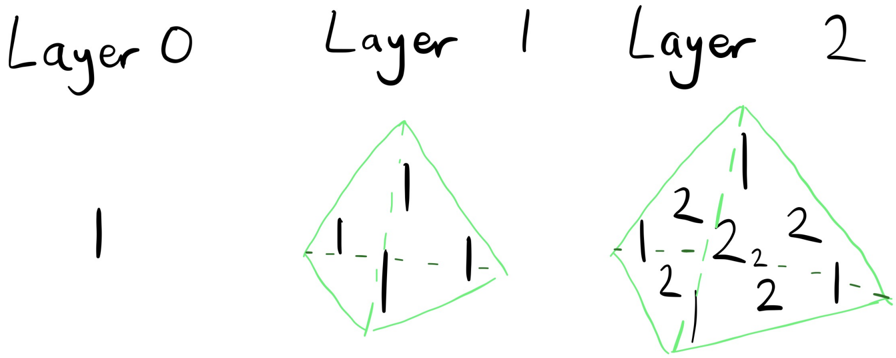

Figure 5: The first 3 layers of Pascal's pentachoron

Beyond 4 dimensions, however, it is not possible to draw even individual layers of the shapes, so geometry can no longer be relied on. What is useful, however, is the trinomial combination formula from before, which can now be extended to any number of variables using the same idea of selecting a_1 elements from a group of n, a_2 elements from a group of $n-a_1$ and so on and so forth.

$$\binom{n}{a_1, a_2, ...a_j} \tag{7}$$

$$= \frac{n!}{a_1![n-a_1]!} \cdot \frac{(n-a_1)!}{a_2![n-(a_1)-(a_2)]!} \cdots \cdot \frac{[n-(a_1)-(a_2)...(a_n-1)]!}{a_n![n-(a_1)-(a_2)-...(a_n)]!} \tag{8}$$

$$= \frac{n!}{(\prod_{i=1}^{j} a_i)! \ (n-\sum_{k=1}^{j} a_k)!} \tag{9}$$

Ultimately, this formula would prove useful for calculating coefficients, especially when the expansions have more than 4 terms.

Finally, considering the 0-dimensional case, Pascal's dot arises: a single one representing the coefficients of a *nullomial* expansion, where no variables are raised to all of the non-negative integer powers. If the blank space below the 1 is taken to show that $0^n = 0$ for all $n > 0$, then this leads to the final trick up Pascal's sleeve: undeniable proof that the value of 0^0 must be 1.