Geometric Visual expression

Generally speaking, mathematics is commonly viewed through a scientific or a statistical lens; a tool to calculate the orbit of planets, quantum probabilities etc. However, in this essay I would like to explore how maths delves into our culture and visual expression using geometry through symmetry and tiling, otherwise known as ‘’tessellations’’. Symmetry and tiling have been developed over the period of 1,400 years, holding mathematical, artistic and historical relevance.The austere brickwork of the Mosque of Ibn Tulun and the geometrical marvel of the mosaics of Alhambra are mere examples of how sophisticated symmetrical geometry can get.Symmetry serves as the bridge between the finite capabilities of man and the infinite mathematical plane. By translating abstract equations into physical geometric patterns, we have turned the abstraction of mathematical equations to an artistic form of abstraction that reflects heritage and history. I see mathematics, not just as a scientific subject but perhaps the ultimate medium of art.

The mathematical roots of symmetry and tiling lie in the group theory and is modelled through invariance which is basically the property of an object remaining unchanged if transformed through translation, rotation and reflection. The set of all such transformations for a given pattern forms a Group(G) under the operation of composition. Any tiled pattern contains a N- fold notation symmetry, to rotate a point by co-ordinate (x,y) by angle θ around the origin, we apply the Rotation Matrix.

$$R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

For a regular 4 fold symmetry which is a square, applying this matrix four times returns the shape to its original position where R^4 = 1 is the identity matrix, forming what is known as a Cyclic Group(C^4). In period tiling, patterns that repeat similar to a chessboard on an infinite plane prove that only 2 to 6 fold symmetries are possible because of the the sum of the diagonal elements of the rotation matrix, also known as the trace which is required to be an integer for a consistent pattern. This is known as the Crystallographic Restriction. This is useful when trying to determine the proper arrangement of materials and to maintain structural integrity as it limits structures such as pavings, mosaics and architectural patterns to triangles ,squares , rectangles and pentagons.

It is also represented through this equation which shows that regular pentagons (n=5) cannot be included in tiling a floor without gaps, otherwise the trace won't be an integer.

$$Tr(R_\theta) = 2\cos\left(\frac{2\pi}{n}\right) \in \{-2, -1, 0, 1, 2\}$$

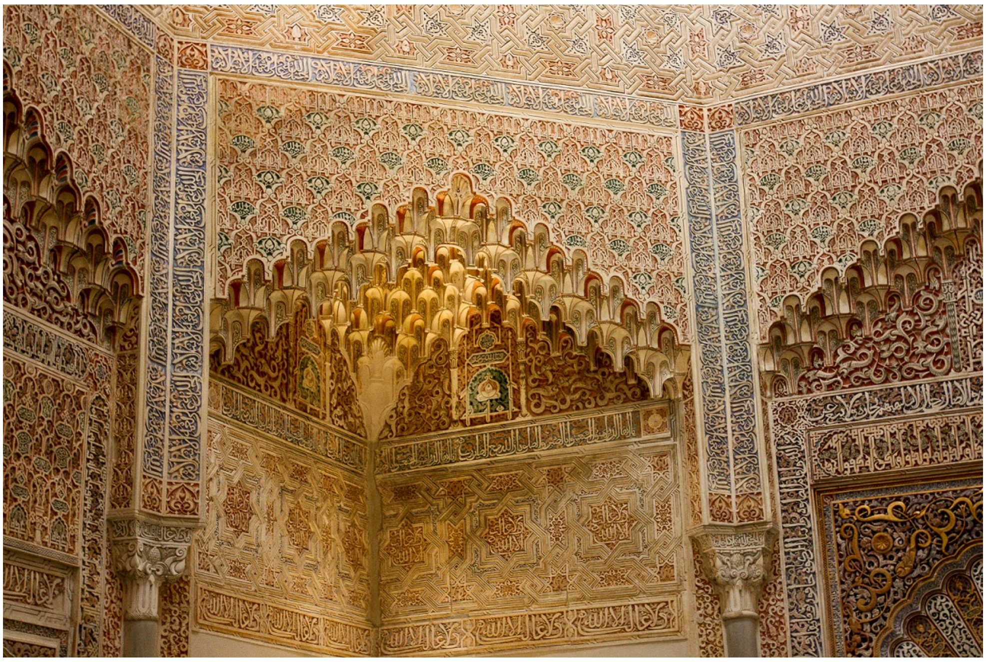
This is the standard way of expressing the trace of a 2D rotation matrix but as mentioned earlier, to satisfy the Crystallographic Restriction Theorem, this trace has to be an integer so to put it more precisely (including 3D rotations), the restriction formula should be as follows:

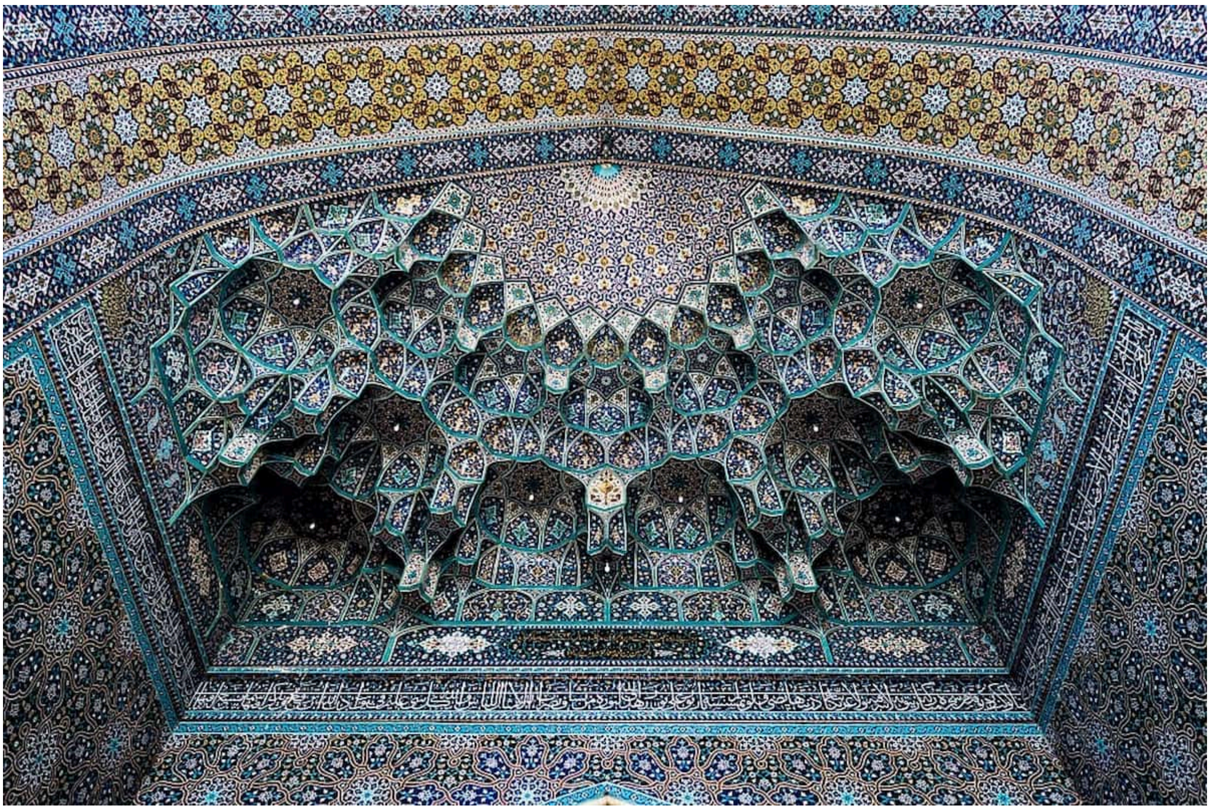
$$Tr(R_\theta) = 1 + 2\cos\theta \in \mathbb{Z}$$

I have mentioned earlier that 5 fold or greater than 6 fold symmetries cannot be possible due to the Crystallographic Restriction but islamic artisans and architects have found a way to bypass this! They managed this because the restriction theorem only applies to patterns that repeat exactly on a translation grid like a chessboard as I said before but in around the 12th century , artisans from the Abbasid period through to the seljuks developed Girih tiles. This is where instead of using one repeating square or hexagon, they used a set of five distinct polygons , decagon , pentagon, hexagon, bowtie and rhombus. Combining these created a staggering 10 fold rotational symmetry but to perfectly cover the plane with no gaps, it must never repeat and since the pattern doesn't repeat, the integer rule no longer applies. Other examples such as the Darbe Imam Shrine which presents a consistent 10 fold pattern as well. The bypass is often found in the Golden Ration because Islamic 10 fold patterns use phi in this formula where $\phi = 1+\sqrt{5}/2$. The distance between these tiles in this instance is not related to integers or whole numbers, but by the power of ϕ which allows a perfect fit.

3D rotations as well as 2D rotations were fundamental to historical architecture. 3D rotations focus on the volumetric symmetry of domes, vaults and Muqarnas which introduces us to another concept, that is the Space Group which is specific to a 3D repeating pattern. It is the sum of three types of symmetry operations which include translations, point groups and compound operations which are completely unique to 3D space. Point groups include the screw axis and the glide planes. A screw axis is a rotation followed by a translation along either x or y axes which is in the famous spiral pattern in the minaret of the Ibn Tulun Mosque while glide planes are a reflection followed by a translation parallel to the reflection plane. An operation in a Space group is written as (R, t) where t is a translation vector and R represents the rotation. If it acts on another point , for example point r then r= Rr+t where r and t are both vectors. Going back to my favourite example of symmetry and tiles which is

Abbasid architecture, the Muqarnas are a great example of 3D rotations as each cell is made so that it fits into a 3D lattice. In the Alhambra palace in Spain, the Muqarnas are organised in radial symmetry which mimics the complexity of 3D Quasicrystalline Space Groups as you can see in these images.





We should now know that the Difference between 2D rotations and 3D rotations is that 3D rotations operate in space groups while 2D operate in wallpaper groups. Wall

paper groups which are 17 apply for the surface decoration or the tiles and 3D space groups that are 230 total apply to the structural volume of the building, columns , domes , vaults and Murqanas. The great architectural innovation here is how the Abbasids seamlessly blended the two while not having the modern technology or computers to simulate these patterns, they operated purely on geometric knowledge and I have gathered research on the methods they have used. Every pattern begins with a circle and its radius establishes the entire unit length for the entire building! Afterwards, the artisan divides the circle’s circumference and this is where they decide the pattern and the symmetry. For 6 fold symmetry, they keep the compass at the same radius and run it around the edges which creates a hexagon shape. For a 5 fold or 10 fold symmetry, they use ϕ (Golden Ratio) contruction as mentioned before.This involves drawing a square inside the circle then finding the midpoint of one side and extending a line to the opposite corner and dragging it to the baseline. This creates a 1: ϕ ratio which could be a perfect pentagon or decagon. The third step would be drawing a sub grid which draws the lines that cross the midpoints of the polygons' edges. The lines must intersect precisely at an angle = 180(n-2)/n
So when the tiles are placed it creates the illusion of an infinite unbroken network as seen in these examples:

The last step is the 3D elevation where the architecture moves to Space Groups which is from the 2D floor to the 3D Muqarnas. The architect applies a Z axis translation which moves the plane upwards. The architect then marks each intersection as a coordinate and applies a Screw symmetry so (x, y, z) = ($R\theta(x,y)$, $z+\triangle z$) where $\triangle z$ is the vertical shift upwards and theta is the angle by which it rotates. When each tier of the Muqarnas is lifted and rotated, it creates the 3D effect that satisfies the 230 Space Group requirements. You can see more on this method from this youtube video: https://youtu.be/pg1NpMmPv48?si=aCRWl3wrI66at3Ah

To conclude, symmetry and tiling isn't just simple geometric calculations, it's about how mathematics can be turned from logical and abstract equations to a soulful visual form of artistic expression. Researching this topic was very enjoyable since i got to mix between these two topics that are thought to be completely distinct when in fact they are not. Ultimately , this geometric art is Group Theory made truly beautiful. By using the Golden Ratio and Screw Symmetries, Abbasid and other pioneers in this geometric art bypassed the restrictions of their era to create such marvels that shouldn’t exist on a flat plane. The Alhambra palace and Ibn Tulun mosque are stone carved with precise and detailed calculations. They are the example that proves that while mathematics is used in scientific fields, it can also be used in complex forms of art.