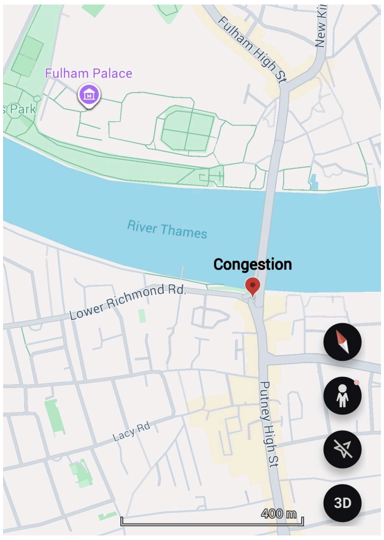
Go With The Flow

Traffic Flow Models

By: Adam Grierson

1 Introduction

Imagine someone pays £1 million just to make your commute to work/school longer. Seems ridiculous to me. Unfortunately this is exactly what happened to me when a £1 million project to reduce congestion on the Putney Bridge junction (that connects the High Street to Lower Richmond Road) backfired, causing severe congestion.[1]



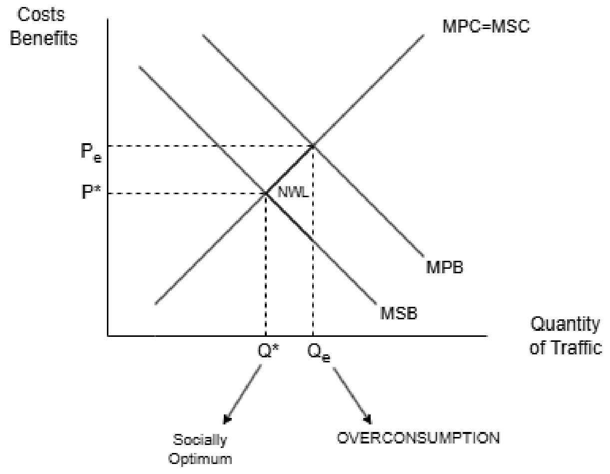
This local problem has driven my interest in traffic flow models (pun intended) and the mathematics behind them, as well as the clear economics link (one of my other A-Level subjects). This essay will explain the mathematical (and some economical) aspects of traffic flow theory through the Lower Richmond Road case study, although it will not be a perfect mathematical model of this junction.

2 Why is it important to improve traffic?

This section of the essay will branch more into economics than maths so I will try and keep it brief but the link between both subjects in this topic is fascinating.

2.1 Negative Externality

Bad traffic is an example of a negative externality in economics. Traffic is 'overconsumed' (weird wording but stick with me). This means that the external costs of traffic (wasted resources, environmental damage) cause a net welfare loss (NWL) to society - meaning that social welfare (or 'happiness') is not as high as it could be. For example: much worse air pollution caused by traffic, could lead to more cases of lung cancer - a cost. A rational government would attempt to decrease consumption of traffic to solve the market failure. A negative externality diagram for the overconsumption of traffic can be seen below, with the NWL represented by the triangle in the middle.



3 Traffic Flow Theory

3.1 Variables

There are three important variables in traffic flow theory:
q is flow, the number of vehicles that pass a fixed point in a given time; veh/hr
k is density, the number of vehicles on a fixed length of the road; veh/km
v is the average velocity of the vehicles; km/hr.

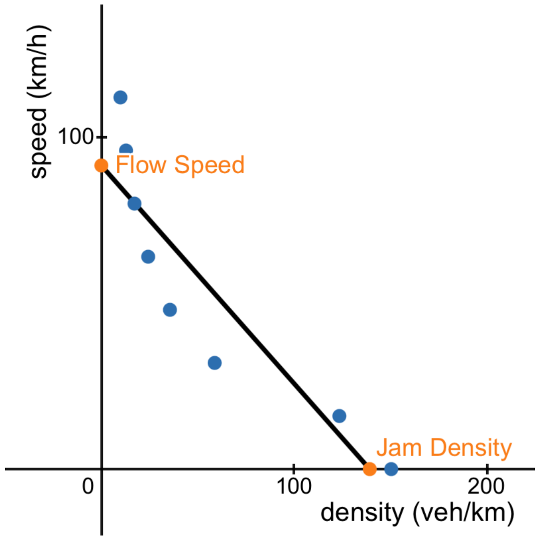
And the equation that ties them all together:

$$q = kv$$

3.2 Speed-density relationship

The relationship between speed and density can be represented through a Greenshield's model (cool surname!), which is important because it models the relationship as a linear one. To give an example of this relationship I utilised a safe headway model (via stopping distances [2]) to calculate density at different speeds.

Which resulted in this relationship:



[with r = -0.904145542]

Another reason the Greenshield's model is so important is because we can use it to find an equation for v in terms of k:
y-intercept = maximum value of v when traffic is free flowing, v_{flow}
x-intercept = value of k when traffic is not moving (cars are 'bumper-to-bumper'), k_{jam}

Therefore the equation of the line in *y* = m*x* + c form is:

$$v = v_f - \frac{v_f}{k_j}k$$

3.3 Flow-density relationship

Substituting the relationship we just found into our equation for q and we find:

$$q = kv$$

$$q = k(v_f - \frac{v_f}{k_j}k)$$

$$q = v_f k(1 - \frac{k}{k_j})$$

A quadratic curve, with a negative parabola - INTERESTING.

And now back to Lower Richmond Road. The theoretical value of v_f is the speed limit, 20mph (=32km/h). And we can estimate k_j with the equation:

$$k_j = \frac{1000}{average\ vehicle\ length\ (in\ metres)}$$

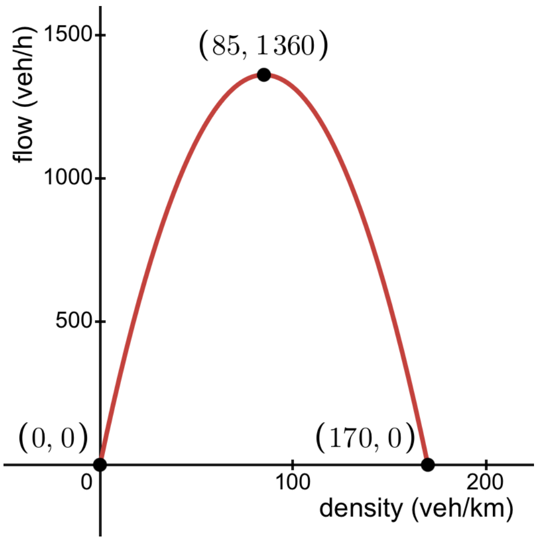
By approximating that 5% of the vehicles are buses and that cars take up 5.5m on average (including a 1m gap) and buses take up an average 12m (including a 2m gap) we find the weighted mean length per vehicle to be 5.825m.

$$k_j = \frac{1000}{5.825}$$
$$k_j \sim 170\ veh/km$$

Substituting into our equation for q:

$$q = 32k(1 - \frac{k}{170})$$
$$q = 32k - \frac{16}{85}k^2$$

Which is shown in this graph.



With simple differentiation we can find q_{max}, the point of optimum speed and density.

$$\frac{dq}{dk} = 32 - \frac{32}{85}k$$
$$0 = 32 - \frac{32}{85}k$$
$$k_{optimum} = 85\ veh/km$$

Hence:

$$v_{optimum} = 36 - \frac{36}{170}(85)$$
$$v_{optimum} = 18\ km/h$$

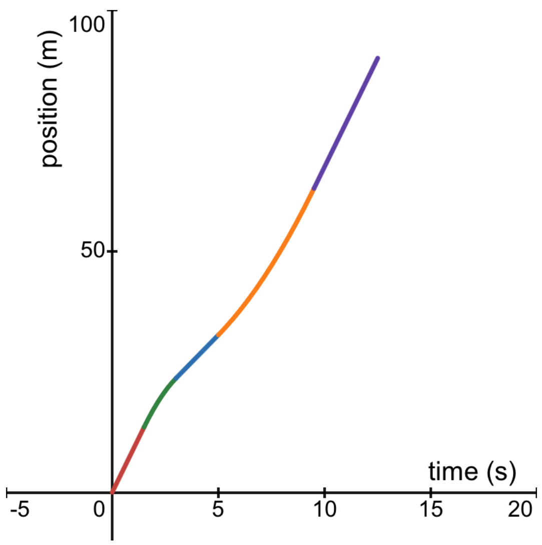
$$q_{max} = 32(85) - \frac{16}{85}(85)^2$$
$$q_{max} = 1360\ veh/hour$$

4 Why is traffic inefficient?

This is the crux of the essay so hold on tight and let's hope I can explain it clearly because much late night thinking went into this section.

4.1 Unnecessary Braking

Lets say I'm walking along Lower Richmond Road on my way to school and decide to cross the road in front of a car, forcing them to brake slightly. The motion of the car as a result of this can be shown on a position-time graph with a piecewise function:



Now let me explain what this graph represents (Warning! High-level colour coding required):

<u>Red</u>
Cruising at 9m/s (=20mph, the speed limit on Lower Richmond Road).

$$y = 9x$$

<u>Green</u>
The driver sees me and slows down to 4.5m/s with constant acceleration of -3m/s^2:

$$s = y \qquad t = x \qquad u = 9 \qquad a = -3$$

$$s = ut + \frac{1}{2}at^2$$

$$y = 9x - \frac{3}{2}x^2$$

<u>Blue</u>
The driver continues at the slower speed while I cross the road.

$$y = 4.5x$$

Orange

The road is clear and the driver accelerates back up to 9m/s with acceleration of 1m/s^2:

$$s = y \qquad t = x \qquad u = 4.5 \qquad a = 1$$

$$s = ut + \frac{1}{2}at^2$$

$$y = 4.5x + \frac{1}{2}x^2$$

Purple

The driver continues to cruise at 9m/s.

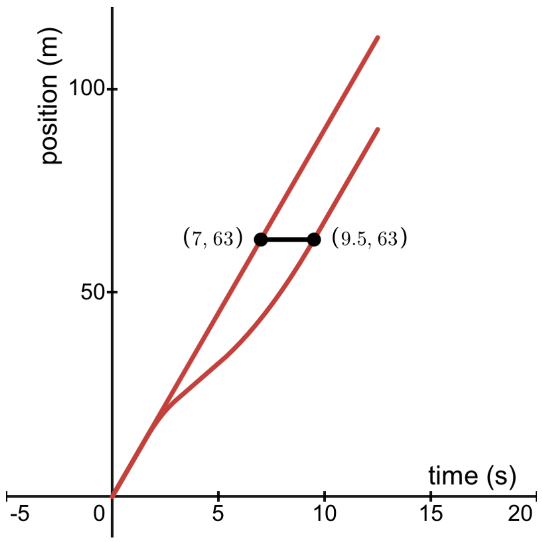
$$y = 9x$$

The difficulty of plotting the graph comes from the fact that each section must be transformed to account for the fact that the subsequent SUVAT equations are not starting at t=0 and to ensure the continuity of the function.

In the end, the piecewise function for the graph shown above is defined as:

$$C(x) = 9x \qquad \{0 \le x < 1.5\}$$
$$= 9x - 1.5(x - 1.5)^2 \qquad \{1.5 \le x < 3\}$$
$$= 4.5x + 10.125 \qquad \{3 \le x < 5\}$$
$$= 32.625 + 4.5(x - 5) + 0.5(x - 5)^2 \qquad \{5 \le x < 9.5\}$$
$$= 9x - 22.5 \qquad \{9.5 \le x < 12.5\}$$

Due to the nature of the position-time graph we can see how many seconds the car lost from braking by finding the horizontal distance between $y = 9x$ and $y = C(x)$ at the point the car returns to cruising speed.



Doing this, we find the driver lost 2.5 seconds (the private cost) because I didn't use a pelican crossing (oops!) and if this occurs multiple times along their journey, this can sum to more significant values of lost time.

4.2 Traffic Shockwaves

Traffic shockwaves [3] occur when one car brakes and the car following behind brakes slightly later due to reaction time and this continues on, overall causing a large sum of cars to brake unnecessarily. Essentially the shockwave is the act of braking that moves backwards through the cars.
On the position-time graph we can add other cars that follow the first one, but brake slightly later due to the driver's reaction time. The next car (c_2) will have equation:

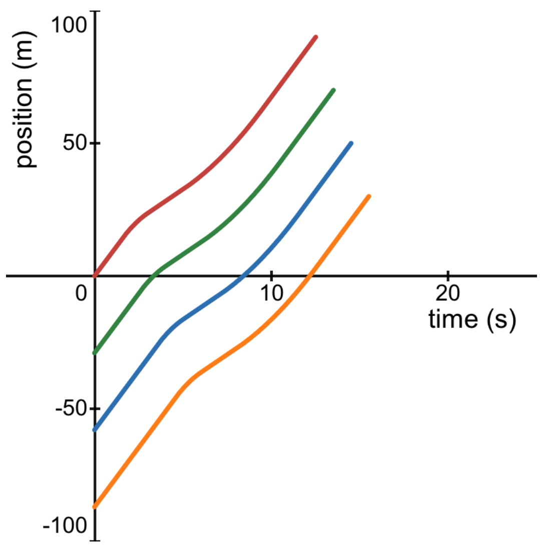
$$c_2(x) = C(x - t_r) - d$$

Where $t_r = reaction\ time$ and $d = initial\ distance\ between\ cars$.

This means that the second car will exactly follow the leading car's movement, just t_r seconds later, having started d metres behind. Similarly, the third car will have equation:

$$c_3(x) = C(x - 2t_r) - 2d$$

Plotting this, with four cars in total results in:



We already calculated the private cost of braking but the traffic shockwave represents the total cost because it's the act of braking passing through ALL the cars on the road.

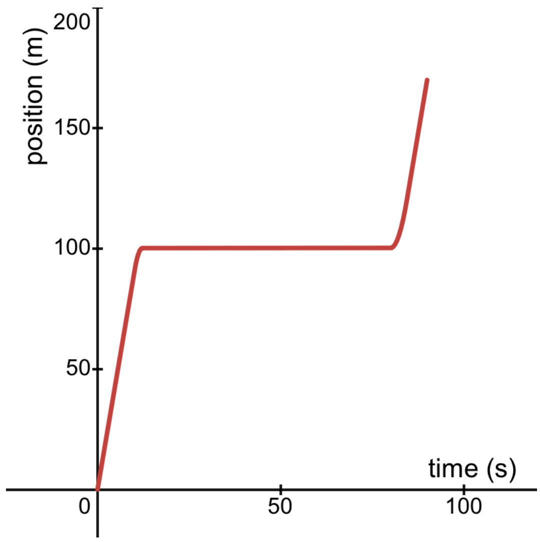
$$total\ cost = private\ cost \times number\ of\ cars$$

For our graph of four cars the total cost is 10 seconds.

4.3 Traffic Lights

Back to the Putney Bridge junction. The main cause of congestion is the traffic light system. The traffic lights were changed to adjust the light cycle based on time of day and congestion. This meant that the traffic light cycle (of 104 seconds [4]) was spending more time on the red light compared to the green light during peak hours for Lower Richmond Road [5]. While on the red light, traffic builds, and the green light does not last long enough to clear the queues so the jam just gets worse and worse.

I intended to produce a position-time graph to display the role shockwaves play at traffic lights but ran into a problem. The distance between cars is not constant meaning the equations of the lines are different for each car, not just a transformation, making it far more complicated to build up the functions of the following cars - SO complicated in fact that I gracefully surrendered my attempt and only managed to formulate an equation for the leading car:



4.3 Braess' Paradox

Another reason that traffic is inefficient is due to the fact that every driver prioritises themself rather than the system as a whole. This self-minded nature of drivers is what causes both results of Braess' Paradox [6], that closing a road can decrease travel times and adding a new route or shortcut can actually increase overall travel times because drivers do not work together to achieve the 'socially optimum' travel time. This is an example of Wardrop's First Principle (another great surname) that traffic will distribute itself so that travel time of all routes will be equal, and there will not be a shorter route that is not in use.

5 How can inefficient traffic be solved?

As mentioned in 2.2, a rational government attempts to solve the overconsumption of traffic - “but how?” I hear you asking.

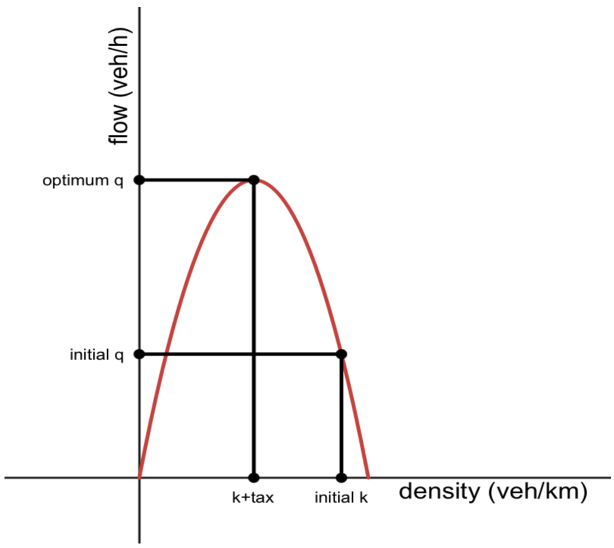
5.1 Solutions

One of the big solutions in London is a congestion charge, which is currently a minimum of £18 per day [7] within the congestion charge zone. The problem with this is that it's regressive pricing (the same for everyone) so targets low income individuals, who suffer the most as a result. Also, a charge on the Putney Bridge junction would not address the actual problem causing jams (the traffic lights).

Not applicable to our case study but smart motorways [8] have many features that help reduce traffic such as variable speed limits, dynamic hard shoulders and traffic management signs.

5.2 Why does tax work?

By imposing a price, the ‘consumption’ of traffic decreases as consumers find other ways to get to their destination for cheaper, like taking public transport. This means the traffic density decreases allowing flow to increase. And if the tax is set perfectly, density decreases to $k_{optimum}$ and flow increases to q_{max} - which can be displayed on our flow-density graph:



6 Limitations: position-time graph

6.1 Acceleration

The values I used for acceleration/deceleration in the position-time graph (1m/s^2 and -3m/s^2 respectively) are typical values for comfortable driving on a relatively slow road. However, it's highly unlikely that every car would have the exact same acceleration or that these values would be constant.
In fact, for a traffic shockwave it's likely the value of deceleration would increase for each next car because they react later so brake more sharply, until eventually a car actually comes to a stop. If my model included this, the total cost would have been greater as the private cost increases for each next car.

6.2 Reaction Time

I modelled reaction time as a constant. An interesting mathematical development would be to model reaction time as a random variable from a normal distribution:

$$t_r \sim N(0.9\,,\ 0.3^2)$$

Where the mean value of 0.9s is from a 1971 reaction time study carried out by Johansson and Rumar [9]. The standard deviation of 0.3s is estimated from the 68-95-99.7 Rule for normal distributions. The only drawback being that a 0s reaction time is not humanly possible but otherwise this change would better reflect real life.

7 Conclusion

Now understanding the complexity of traffic flow models I feel slightly less bitter about my longer commute to school along Lower Richmond Road, despite the economic significance of improving traffic. And I certainly won't be crossing the road in front of a car anymore, although I am impressed that using the position-time graph allowed me to quantify the cost of this action given this is usually a huge difficulty in economics. Once again, I think the bridge between maths and economics in this topic is so interesting (and yes, that pun absolutely was intended).

8 Sources and Further Information

[1] More information on Putney Bridge case study
https://southlondonnews.co.uk/local/wandsworth/putney/putney-bridge-1m-redesign-disaster-wandsworth-congestion-crisis/
Last accessed: 12/04/2026

[2] Stopping distances at different speeds
https://www.theaa.com/breakdown-cover/advice/stopping-distances?msockid=06683f799c8a69761ab82a599d31687f
Last accessed: 12/04/2026

[3] A more detailed mathematical explanation of traffic shockwaves
https://tomrocksmaths.com/2023/02/16/traffic-shock-waves/
Last accessed: 12/04/2026

[4] Traffic light cycle of 104 seconds
https://putney.news/2026/02/10/council-buries-the-truth-even-with-fixes-traffic-is-twice-as-bad-with-new-junction/
Last accessed: 12/04/2026

[5] Lower Richmond Road losing green light time
https://putney.news/2025/11/14/putneys-million-pound-junction-disaster-how-approved-plans-went-wrong/
Last accessed: 12/04/2026

[6] Braess' Paradox explained further
https://www.youtube.com/watch?v=ZiauQXIKs3U
Last accessed: 12/04/2026

[7] Congestion charge information
https://tfl.gov.uk/modes/driving/congestion-charge
Last accessed: 12/04/2026

[8] Smart motorway information
https://www.theaa.com/driving-advice/smart-motorways?msockid=06683f799c8a69761ab82a599d31687f
Last accessed: 12/04/2026

[9] Johannson and Rumar findings
https://psycnet.apa.org/record/1971-21979-001
Last accessed: 12/04/2026