Group theory, Lie groups and Modular forms

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April 2026

4 EULERS FORMULA, ROTATION AND SYMMETRY

Inspired by 3blue1browns very first video of introduction to group theory via Eulers formula I think that video very much sets the ground work for the theory of lie groups in fact that very video featured a lie group which is the group of continuous symmetries of a circle and a lie algebra which is simply the line 1+xi where x is real under a lie bracket and the exponential map as mentioned in the video acted as a homomorphism between the additive group of real numbers and the multiplicative group of complex numbers a similar idea can be used when we consider the lie algebra 1+xi as a additive group and the exponential map acts as a homomorphism between this additive group and the rotation symmetry group of a circle and the complex numbers with absolute value 1 under multiplication every lie group is a vector space so is a group under vector addition so every lie algebra is an additive group

Groups have a much more abstract definition but groups are essentially a set describing symmetries of another set a group G acts on a set A if G is the set of all maps of A that only permute the elements of A and leave the set the same so the group of symmetries of a circle by rotation by any angle where the center of rotation is the center of the circle is called the group action of the set of points on the circle however one thing we may notice is that if you combine 2 symmetries you get another symmetry so the set of symmetries to has closure under the operation of combining symmetries also there is a symmetry of simply doing no change to any element of the set and this is called the identity symmetry which has the property such that if you have an element of the group g , A then AI=IA=A where I is the identity symmetry of doing nothing also if A is a symmetry then A*S=S so if we apply A$^(-1)$tobothsideswhereA$^(-1)A = I$whereA$^(-1)$isnotnecessarilythemultiplicativeidentityweget(A$^(-1)) * A * S = (A^(-$

$1))SsoI*S = (A^(-1))SsinceIisanidentitysymmetryofthegroupGalltheelementsremainunchangedsoS = (A^(-1))SsoA^(-1)isalsoasymmetryofSsoforeveryAinGA^(-1)isinG$

5 GROUPS, VECTOR SPACES AND ALEGBRAS

a group is defined as an orderd pair (S,*) where S is a set and * is an operation that combines symmetries which could be addition for example when looking at translational symmetries of a real number line as translation in the real number line is described as adding a real number so if you have translations by 2 real numbers then addition of the 2 real numbers corresponds to the translation of by the sum of the 2 real numbers the group satifies 4 axioms being for any a,b in S a*b is in S (closure) for a,b,c in S (a*b)*c=a*(b*c) (associativity). There exists I in S such that for every x in S Ix=xI=x (Identity). For every a in s there exists b in S such that a*b=I. a field F is defined with the following properties:

(F,+) is an abelian group (abelian meaning the group operation is commutative over the set)

(F,x) is an abelian group where the identity of (F,+) doesnt equal the identity of (F,x)

a vector space is an orderd pair of a set V and a field F with the property such that (V,+) is an additive group, there exists an operation called sczlar multiplication denoted by * such that for any b in F b*v is in V for v in V also let the multiplicative identity of F be 1 then 1*v=v for any a,b in F (a+b)v=av+bv also v(a+b)=va+vb given v is in V. An alegbra over a field f A is a vector space with field of scalars F such that there exists a product operation x.y=f(x,y) where

this operation is both left and right distributive and aT.bC=(ab)(TC) where a b are in f and T C are in A

6 HOMOMORPHISMS AND ISOMORPHISMS

2 groups (A,*) and (B,) are homomorphic if there exists a function f that maps A to B or B to A such that f(a*b)=f(a)f(b) for a,b in A if the function maps A to B and f(ab)=f(a)*f(b) for all a b in B if f maps B to A such a function is called a homomorphism between A and B a linear map between (V,F) and (V2,F2) is a homomorphism function f between (V,+) and (V2,+) that has the property f(av)=a*f(v) where * is scalar multiplication given a is in F v is in V if f maps v to v2 but given a is in f2 v is in v2 if f maps v2 to v. 2 groups (A,*) (B,) are isomorphic if the homomorphism is a bijective function (maps elements one to one hence is invertible)

7 KERNALS,QUOTIENT GROUPS AND COSETS

(Since order of multiplication matters you could assume we are talking about either left or right cosets here and the proof still holds) If you looked at modular arithmatic you have allready seen cosets, cosets are essentially classes of elements with something in common so produces similar results under the group operation for instance in modular arithmetic we look at objects like [3]$_1 2 which is 3 modulo 12 and we have operations between these cos$
$[9]_1 2 = [0]_1 2 but each of these cosets represents an entire set of integers for instance [3]_1 2 is the set ..., -27, -15, -3, 0$
$(or (N, +)) is a group and the cosets of the group (which in this case are just the elements [1]_N [2]_N etc in a group G the cos$
$is an operation so that cosets (aH) * (bH) = (ab) H the kernal of a homomorphism f :$
$G -> G2 Is a where f(a) = I where I is the identity of G2 given a linear map f : V1 ->$
$V2 where f(v) = M v the kernal of f is the set of all vectors that get mapped to the additive identity (Identity of (V2, +)$
$V1 -> V2 is just the kernal of the homomorphism f : V1 -> V2 where (V1, +)(V2, +) are groups by definition of$
$V1 -> V2 is a homomorphism by definition of linear maps$

8 ORDER FROM CHAOS (WHERE ALGEBRA MEETS CALCULUS)

Lie groups are all about the idea of continuous symmetry. A lie group is a group that is also a differentiable manifold a single definition that bridges calculus and algebra. A differentiable manifold is a set of points which in not rigorous terms is a set of points that as you zoom in become more closer to a homomorphism of a vector space allowing differentiation and tangent spaces at each point if we have differentiable manifolds over C that the tangent space may be more than 1 dimentional. The formal definition of the derivative of f at x is given in a similar way as epsilon delta for limits so for a sequance z0,z1,z2,... where for any given epsilon there is a delta such that abs(z(delta)-x)¡epsilon then f'(x) is a value such that for every epsilon there is a delta such that abs((f(z(delta+1)-f(z(delta))/(z(delta+1)-z(delta))¡epsilon this value may or may not exist. Continous symmetries such as the rotational group of a circle seem to have order wheras some objects which have a discrete set of symmetries have more chaos like a clock modulo N where there is less continuity between the symmetries but as turns out with objects like the e8 lattice that continous symmetries are the key to uncovering ideas in discrete mathematics

9 CONTINOUS SYMMETRIES AND EXPONENTIAL MAP

we can define exp(x) at the value of g(t) when t=1 where g(t)=exp(xt) this definition allows us to think of the process of exponentiation rather than how points under the image of an exponential map relate to their neighboorhood points just by doing a seemingly not expected move that doesnt seem to do much. Exp is a homomorhpism between (A,+) and (exp(A),x) where exp(A) is the set of all elements of A after applying exponentiation to each element. (d/dt)*g(t)=d/dt*exp(xt)=x*d/dt*exp(xt) so in dynamics terms we have a vector moving around a geodesic with velocity equal to x times the total displacement which changes with x

hence cauing a geodesic curve path. However as you notice we need some kind of innitial condition which is g(0)=1 so we can immediatly get the exponential at 0. If you have a additive group which is the tangent space to exp(A) at 1 (the identity of (exp(A),x)) so the intersection between exp(A) and the tangent space is 1 then the preimage of the exponential map (input set) is this additive group generally speaking so (A,+) is an additive group which is also the tangent space of (exp(A),x) however A is also a vector space using properties of exponentiation (proof left as an exercise to reader)

10 TANGENT SPACE AND CR EQUATIONS

The tangent space of a differentiable manifold S at x is the set of all possible values of f'(x) where f maps R to a 1d differentiable manifold subset of S. For a differential function over R g:R-¿R you may think that the tangent space of D where D=(x,g(x)) for x in R at a point is just the derivative of the function at the point but you have to consider the fact that g'(kt) where k is real are all functions that map the real numbers to the points on D and g'(kt)=k*g'(kt) which are all possible scalar multiples of g(kt) with field of scalars R so even for a 1 dimensional differential manifold the tangent space isn't just the derivative of the function at that point but rather an entire vector space line. For complex functions we describe the derivative of the function at a point using a jacobean matrix and the cauchy rieman equations which gives the tangent space vector space a corresponding matrix which maps $\mathbf{R}^2 to that tangent space these matricies are used to derive the cauchy rieman equations for complex differen$

LIE ALGEBRAS AND THE LIE BRACKET

A lie algebra is defined as an algebra where the product operation is a lie bracket [x,y] which satisfies certain axioms including the jacobi identity where [x,[y,z]]+[y,[x,z]]+[z,[x,y]] [x,y] is often represented as $\mathbf{ad}_x(y) and the motivator for making lie algebras have to satisfy this identity is that we want ad_{[a,b]}(x) = [ad_a(x), ad_b(x)] so that the adjoint function preserves the lie bracket and this can be used to show the jacobi identity als$
$1) where g h are in G for a group G and (wont be mentioned here sorry due to time pressure) congugation in the lie group$

11 MODULAR FORMS

brief mention here that modular forms are function in simple terms invariant under symmetries (az+b)/(cz+d) where abcd are integers (oversimplified) and it turns out that these functions form a group under composition and for automophic forms this group is lie also

the coefficents of fourier series of modular forms have connections to both the theory of finite groups and number theory