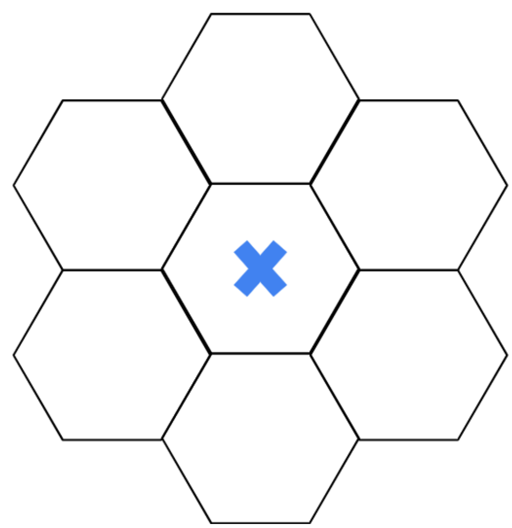
How hexagons managed to ruin my life for 3 weeks in the best way possible.

Introduction:

One day on the bus ride to school, a problem popped into my mind like divine inspiration which I thought would be simple. Over the course of the next few days, I spent at least 7 hours trying to solve this which led to nowhere. It was then I decided that this would be my entry to this competition, so of course fate decided to play a prank on me. This is simultaneously the hardest and the most fun problem I have attempted.

The problem:
Given a hexagonal tiling as seen in the picture.
And that there is a x in the center.
Each turn, there will be a probability that an empty neighbouring cell becomes 'infected' with an x.

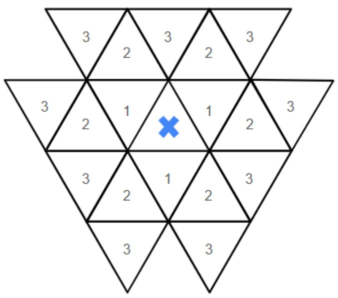


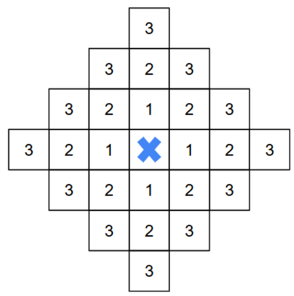
This probability is determined by the amount of neighbouring x the cell contains.
If there are 1 neighbouring cell containing an x, the probability is ⅙.
If there are 2 neighbouring cells containing an x, the probability is ⅓.
If there are 3 or more neighboring cells containing an x, the probability is ½.

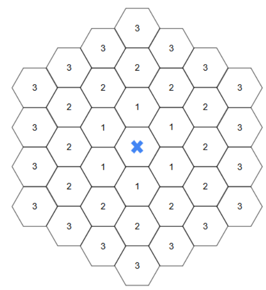
What is the average amount of turns it takes until the grid is filled?

Why hexagons?

Out of all the regular tiling polygons, hexagon is the only one that creates this complex problem, as there are at least 3 neighbours connected to a cell at all times. In other shapes, this only happens when it is 2 or 3 layers deep, while hexagons present this problem immediately. Here, 1 layer creates probabilities that are dependent on each other which is different to every other regular tiling polygon. Hexagons create the most complicated and the most interesting problem.



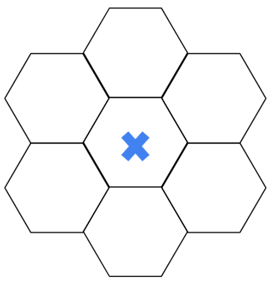




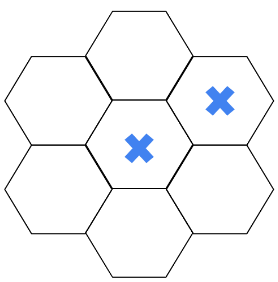
States

Due to the multiple neighbouring cells with a different probability, there are more cases that need to be mapped out for different amounts of x on the grid. As such I will be referring to the states with the variables a,b and c. Which states the amount of cells with the probability of a, b, and c. In which a is ⅙, b is ⅓, and c is ½. For example the starting position is 6a.

Starting position:
Situation 1: 6a

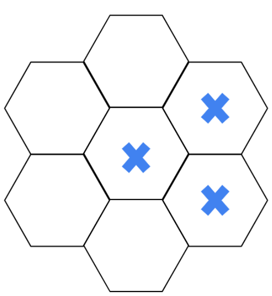


2x:
Situation 1: 3a + 2b

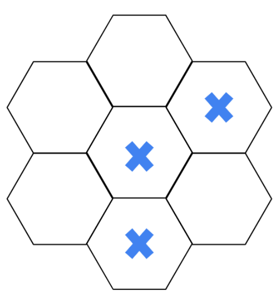


3x:

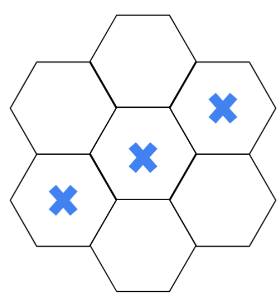
Situation 1: 2a + 2b



Situation 2: a + 2b + c

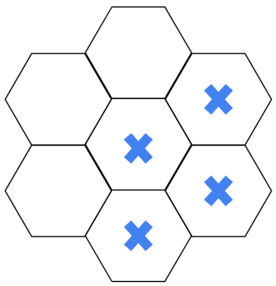


Situation 3: 4b

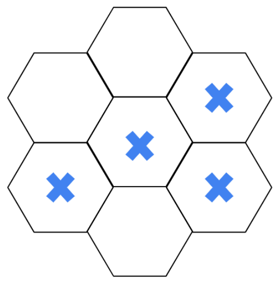


4x:

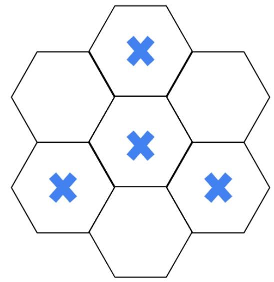
Situation 1: a + 2b



Situation 2: 2b + c

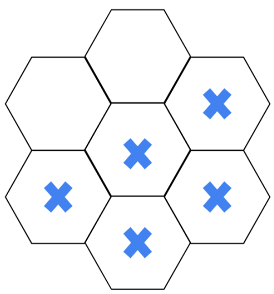


Situation 3: 3c

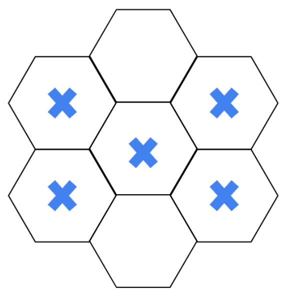


5x:

Situation 1: 2b

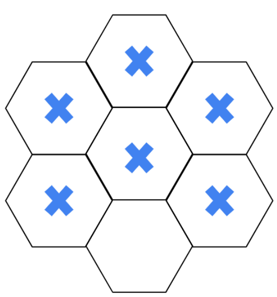


Situation 2: 2c



6x:

Situation 1: c



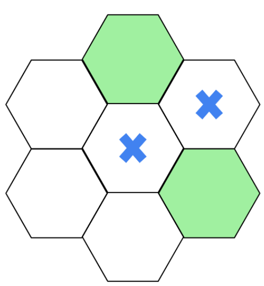
Probability of transitioning states

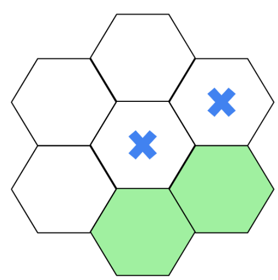
After creating those states, I worked out the branches of probability from one state to another. For the benefit of my slowly decreasing mental state, I assign the states as S_{xy} where x and y indicate which state it is, x is the amount of filled out x while y is the situation number. For example, S_{33} is the one with x forming a straight line.

Each transition is calculated with only 1 turn which is the possibility that state a would become state b.
The $P(S_n \rightarrow S_m)$ is dependent on 3 things:
The probability of empty cells which is required to become x
The probability of empty cells remaining empty
The amount of combination(s) that could form S_m

For example with $S_2 \rightarrow S_{41}$,
There are 2 possible ways to transform S_2 into S_{41} which again would be best demonstrated with pictures.

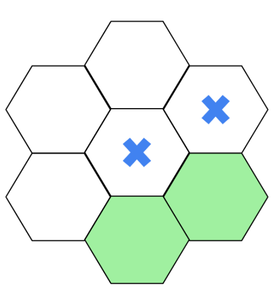
2b becomes x or 1a and 1b becomes x

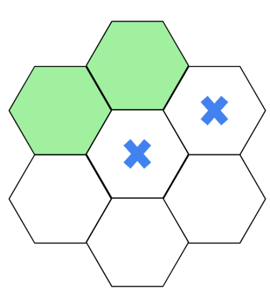


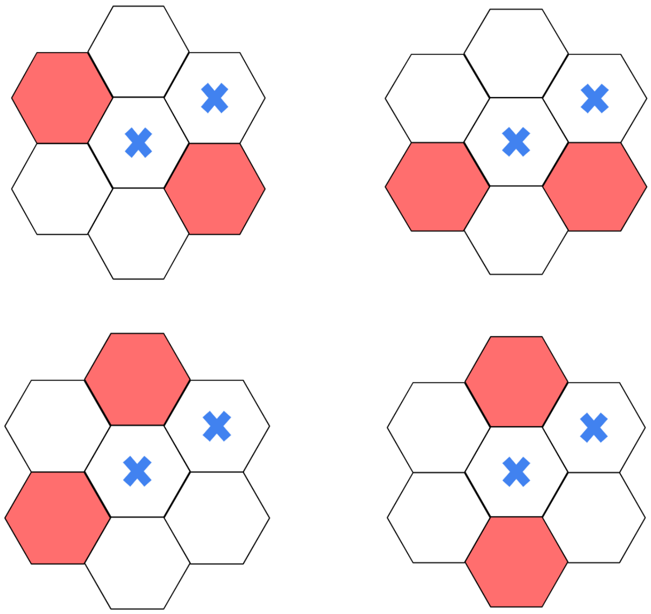


Calculating the first situation is a lot simpler than the second, as there is only 1 possible way that it could occur.
$(\frac{1}{3})^2 \cdot (\frac{5}{6})^3 = 125/1944$

For the second situation, it is slightly more tricky.
The following are all possible states with 1 cell of a and 1 cell of b that have turned into x, and green indicates it would form S_{41}.







As such the probability of this would be:
$\frac{1}{6} \cdot \frac{1}{3} \cdot (\frac{5}{6})^2 \cdot \frac{2}{3} \cdot 6 \cdot \frac{1}{3}$ or $\frac{1}{6} \cdot \frac{1}{3} \cdot (\frac{5}{6})^2 \cdot \frac{2}{3} \cdot 2 = 25/486$

While I did manually find out every single transitioning state's probability, I would not be writing them all out as it is repetitive and similar to the process shown above, and also to not bore my dear readers. For those who are trying this at home, the process is not hard, just tedious (This would become a trend).

Even though I found all of these branches of probability for a method which both did not work and tricked me into thinking this problem is a lot simpler than in reality. The foundation created here was instrumental later on, and I am most certainly grateful that past me suffering through the tedious nature of it.

Stuck in a loop

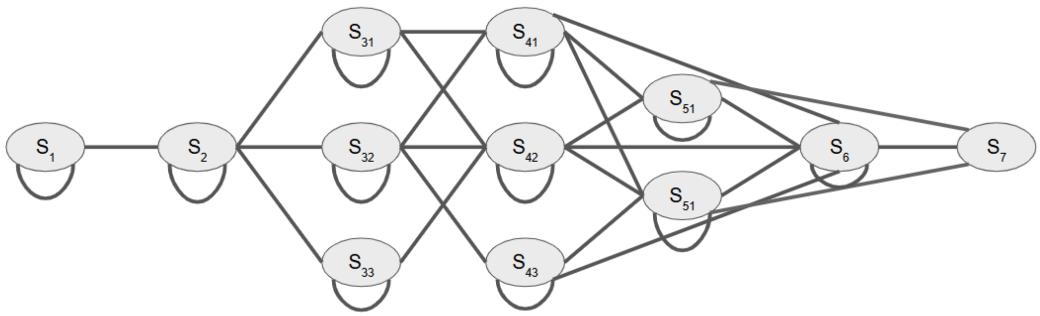
I quickly hit another problem… I have no idea what to do. I was in an infinite series of summation. To best explain this, I would have to use more algebra, p(x) and q(x), where p(x) is the probability of progress, and q(x) is the probability of looping. Looping here is when no progress is made, since there is no guarantee that any empty cell would become x, it is possible for every single cell to roll 'not to turn into x'. As such, even in the simplest case of S_6 with only a ½ chance of proceeding, it is still an infinite summation, of $p(x) + q(x) \cdot p(x) + q(x)^2 \cdot p(x) + q(x)^3 \cdot p(x) + \ldots$ which is 2 from geometric series. Though I could do this for all the states, the ones after this would be exponentially harder; I am going to find another way.

It was as if fate was personally looking after me, Veritasium's video, "The Strange Math That Predicts (Almost) Anything", showed up on my recommendation page. From that, I learnt about Markov chain. Then I looked up the wikipedia page of Markov chain because I thought it would be relevant to this, and I found this line "A state i is called absorbing if there are no outgoing transitions from the state." This is my first interaction with absorbing Markov chain.

I began to make progress once again after being stuck for so long.

Expected progress

From there, I drew out a state transition diagram to give me an idea of what I was working with.



Not all the branches are drawn on this diagram.

Normally, this would be the beginning of a Matrix nightmare. But I chose a different path, one which is arguably worse (I hate you past me). I used recursion, and found every single state.
The average amount of steps is:

$$E(S_n) = 1 + \sum P(S_n \to S_m) \times E(S_m)$$

This is calculating the expected step of S_n by summing all the branches of possibility. Similar to that of an expected value of a normal probability, this is just finding the probability of transitioning states going to all other possible states. The 1 is added due to the fact that it is taking 1 turn, as even if it went from S_n to S_n it still requires a turn for that action. It is best to understand this with an example.

Let say we are calculating expected steps of S_6
$E(S_7) = 0$ as there are no remaining steps need once we reach there
$E(S_6) = 1 + P(S_6 \to S_6) \cdot E(S_6) + P(S_6 \to S_7) \cdot E(S_7)$
$E(S_6) = 1 + P(S_6 \to S_6) \cdot E(S_6) + P(S_6 \to S_7) \cdot 0$
$E(S_6) = 1 + P(S_6 \to S_6) \cdot E(S_6)$
$P(S_6 \to S_6)$ is ½ as there is a ½ chance for it to not become x in a turn
$E(S_6) = 1 + \frac{1}{2} \cdot E(S_6)$
$E(S_6) = \frac{1}{2}$

Now, lets calculate the expected steps of S_{51}
$E(S_{51}) = 1 + P(S_{51} \to S_{51}) \cdot E(S_{51}) + P(S_{51} \to S_6) \cdot E(S_6) + P(S_{51} \to S_7) \cdot E(S_7)$
Plugging in the numbers,
$E(S_{51}) = 1 + 2 \cdot \frac{1}{3} \cdot \frac{2}{3} \cdot E(S_{51}) + 2 \cdot \frac{1}{3} \cdot \frac{2}{3} \cdot 2$
$E(S_{51}) = 17/5$

The final results

As you could probably guess with this paper, I did all the calculations manually… for all the states with recursion. I promise I have not gone insane yet.

So without further ado, here is the result I found.

States	Expected Progress
S_7	0
S_6	2
S_{51}	17/5 (=3.4)
S_{52}	8/3 (~2.67)
S_{41}	226/81 (~2.79)
S_{42}	56/15 (~3.73)
S_{43}	22/7 (~3.14)
S_{31}	8779441/1542240 (~5.69)
S_{32}	21095/5236 (~4.03)
S_{33}	859/195 (~4.41)
S_2	91816716599/15923011104 (~5.77)

Enough stalling, if you are still here you probably want to know the answer.
Drumroll, please!

The expected steps from S_1 to S_7 is:
$E(S_1) = 1 + P(S_1 \rightarrow S_2) \cdot E(S_2) + P(S_1 \rightarrow S_{31}) \cdot E(S_{31}) + P(S_1 \rightarrow S_{32}) \cdot E(S_{32}) + \ldots + P(S_1 \rightarrow S_6) \cdot E(S_6) + P(S_1 \rightarrow S_7) \cdot E(S_7)$

After plugging in the numbers, we get
21472427494159/3167352292104
Or 6.779 (to 3dp)

Final thoughts

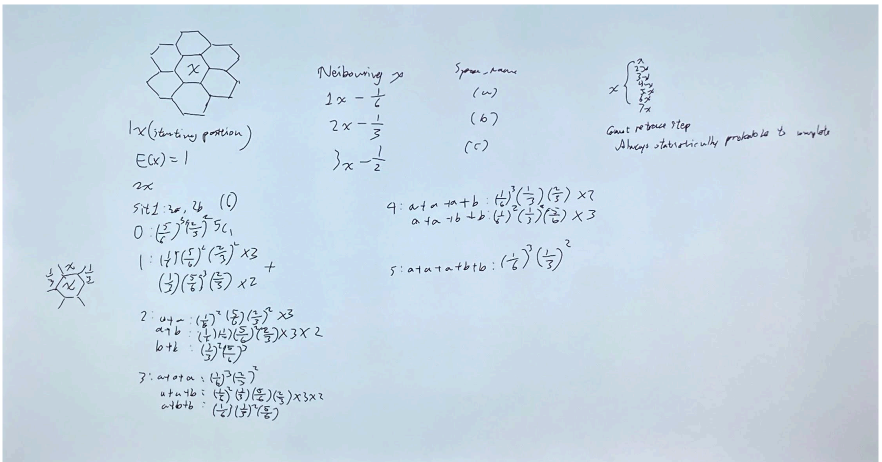
Through this journey of statistics, probability, and hexagons; I laughed, I learnt, and I fell deeper in love with Maths. There are many different ways to approach a problem, I just happen to find one of the most tedious paths. But ultimately I'm happy and glad that I persevered. After 3 weeks (of pain and suffering) I managed to solve this problem, which was a lot harder than it first appeared. I came across the Markov chain, something new and exciting. Hopefully I will see you soon, I am here waiting.

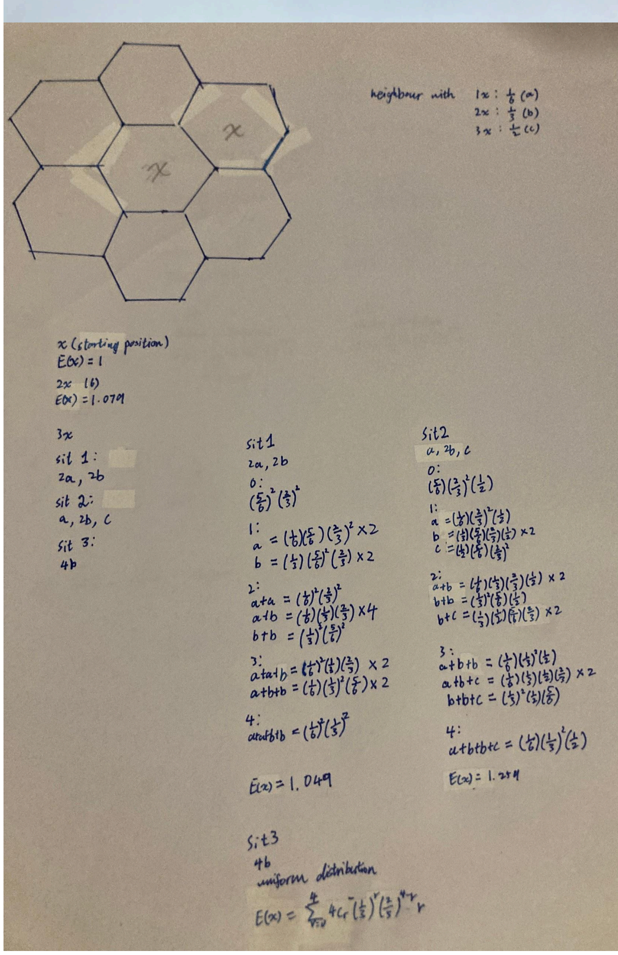
As for Hexagons, thank you, CGP Grey was right, hexagons are the bestagons.

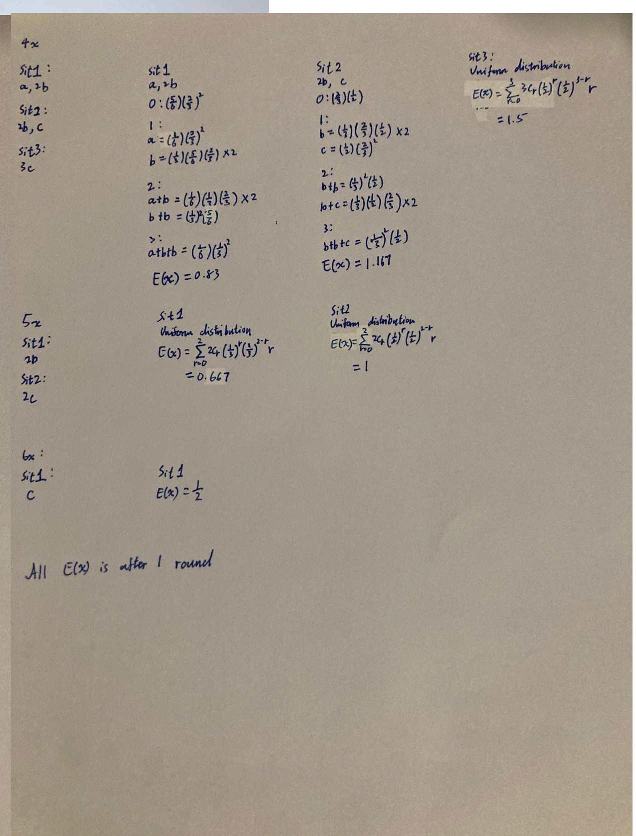
Appendix

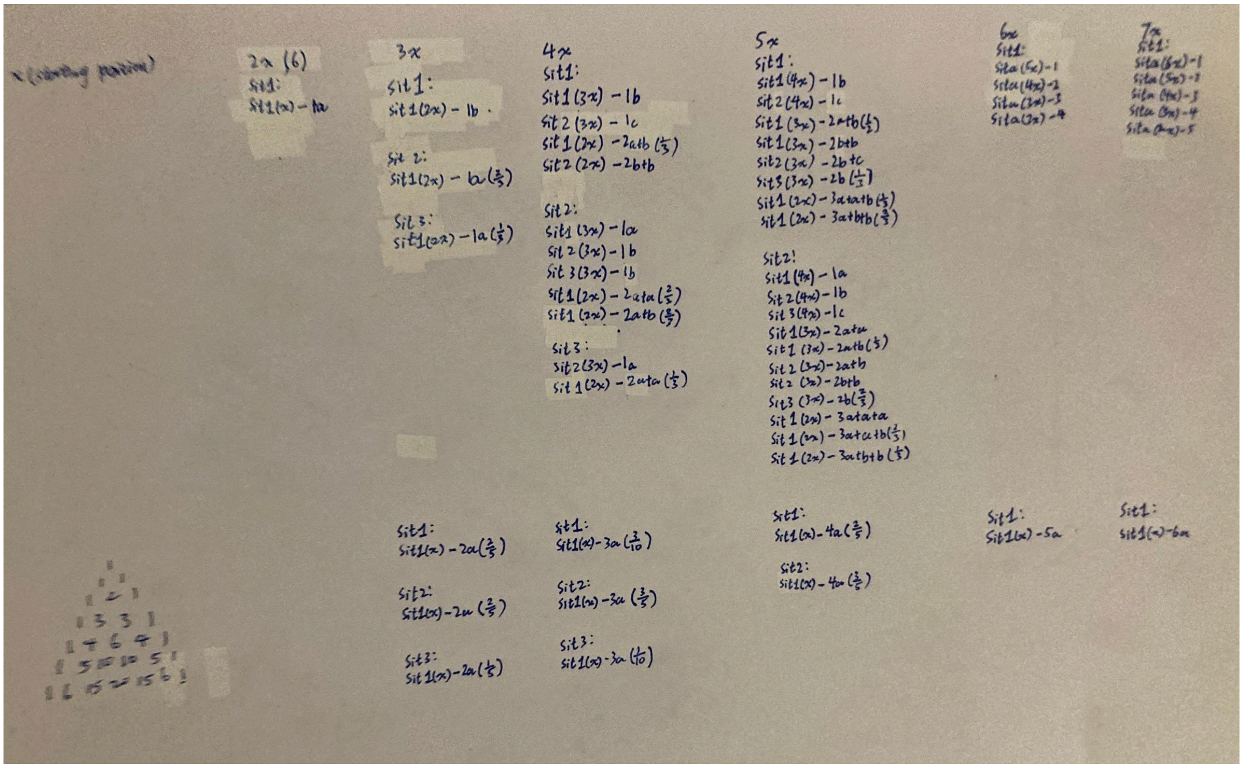
These are all my working out for the Math above (Sorry for Picture quality in advance)

Transitioning probability from S_n to S_m:









For a little explanation of this expression:
Sit1(2x) - 1a(⅔)

Sitn refers to the situation I am using, where n is indicating which one.
Sita means any situation in the situation.
(2x) refers to the starting amount of x .
1a refers to which branch of probability I am using.
(⅔) refers to the additional probability needed in the branch.

Expected steps:

$S_7 = 0$

$S_6 = 2$

$S_{51} = \frac{17}{5}$

$S_{52} = \frac{8}{3}$

$S_{41} = \frac{226}{51}$ 4.37

$S_{42} = \frac{56}{15}$

$S_{43} = \frac{22}{7}$

$S_{31} = \frac{8779441}{1542240}$ 5.6927

$S_{32} = \frac{21095}{5236}$ 4.0288

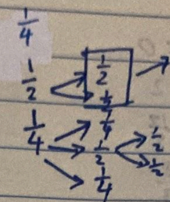
$S_{33} = \frac{859}{195}$ 4.405

$S_2 = \frac{91\,816\,716\,599}{21\,936\,519\,104}$ 5.766

$S_1 = \frac{52\,422\,497\,899\,193}{8\,528\,161\,850\,208}$ 6.149

$\frac{1}{2} + \frac{1}{4} \cdot 2 + \frac{1}{8} \cdot 3 \ldots$ Infinite summation as $x \to \infty$

$S_6 \to S_7$

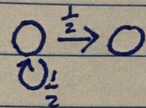


$\sum_{r=1}^{n} \frac{1}{2^{r-1}} r$

$E_{s_6} = 1 + \sum_{r=1}^{n} P(x=r) \cdot E(r)$ steps

$E_{s_7} = 0$ as no more steps

$E_{s_6} = 1 + \frac{1}{2} E_{s_6} + \frac{1}{2} E_{s_7}$



$\sum_{r=0}^{n} \frac{1}{4^r} \left(\frac{1}{4}r + \frac{1}{2} \sum_{s=0}^{m} \frac{1}{2^s} s \right)$

$E_{s_6} = 2$

$E_{ss1} = 1 + \sum_{r=1}^{n} P(x=r) \cdot E(r)$ steps

$\frac{1}{4} \cdot 1 + \frac{1}{2} \cdot$ sum

$= \frac{20}{9}$

≈ 2.22

$E_{ss1} = 1 + \left(\frac{2}{3}\right)^2 E_{ss1} + 2\left(\frac{1}{3}\right)\left(\frac{2}{3}\right) E_{s_6} + \left(\frac{1}{3}\right)^2 E_{s_7}$

$E_{ss1} = 1 + \frac{4}{9} E_{ss1} + \frac{4}{9} E_{s_6} + \frac{1}{9} E_{s_7}$

$E_{ss1} = 1 + \frac{4}{9} E_{ss1} + \frac{8}{9}$

$\frac{5}{9} E_{ss1} = \frac{17}{9}$

$E_{ss1} = \frac{17}{5}$

≈ 3.4 steps

$\frac{4}{9}$ loop

$\frac{4}{9}$ progress

$\frac{1}{9}$ end

Python $= 4.42$

$E_{ss2} = 1 + \left(\frac{1}{2}\right)^2 E_{ss2} + 2\left(\frac{1}{2}\right)^2 E_{s_6} + \left(\frac{1}{2}\right)^2 E_{s_7}$

$E_{ss2} = 1 + \frac{1}{4} E_{ss2} + \frac{1}{2} E_{s_6} + \frac{1}{4} E_{s_7}$

$\frac{3}{4} E_{ss2} = 2$

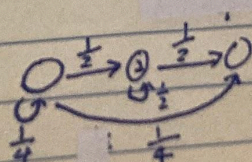
$E_{ss2} = \frac{8}{3}$

≈ 2.67

$\frac{1}{4}$ loop

$\frac{2}{4}$ progress

$\frac{1}{2}$ end



S_{41}:

$\left(\frac{5}{6}\right)\left(\frac{2}{3}\right)^2 S_{41}$

$2\left(\frac{1}{3}\right)\left(\frac{5}{6}\right)\left(\frac{2}{3}\right) S_{51}$ $\quad \frac{34}{27}$

$\left(\frac{1}{6}\right)\left(\frac{2}{3}\right)^2 S_{52}$ $\quad \frac{16}{81}$

$2\left(\frac{1}{6}\right)\left(\frac{1}{3}\right)\left(\frac{2}{3}\right) S_{61}$ $\quad \frac{1}{3}$

$\left(\frac{1}{3}\right)^2\left(\frac{5}{6}\right) S_{61}$

$S_{41} = \frac{10}{27} S_{41} + \frac{226}{81}$

$\frac{17}{27} S_{41} = \frac{226}{81}$

$S_{41} = 4.43$

S_{42}

$\left(\frac{2}{3}\right)^2\left(\frac{1}{2}\right) S_{42}$

$\left(\frac{1}{2}\right)\left(\frac{2}{3}\right)^2 S_{51}$ $\quad \frac{34}{45}$

$\left(\frac{1}{3}\right)\left(\frac{2}{3}\right) S_{52}$ $\quad \frac{16}{27}$

$\left(\frac{1}{3}\right)^2\left(\frac{1}{2}\right) S_{61}$ $\quad \frac{3}{9}$

$2\left(\frac{1}{3}\right)\left(\frac{1}{2}\right)\left(\frac{2}{3}\right) S_{61}$

$S_{42} = \frac{2}{9} S_{42} + \frac{542}{135}$

$\frac{7}{9} S_{42} = \frac{542}{135}$

$S_{42} = \frac{56}{15}$

S_{43}:

$\frac{1}{8} S_{43}$

$\frac{3}{8} S_{52}$ $\quad 1$

$\frac{3}{8} S_{6}$ $\quad \frac{3}{4}$

$\frac{7}{8} S_{43} = \frac{11}{4}$

$S_{43} = \frac{22}{7}$

S_{31}

$\left(\frac{5}{6}\right)^2\left(\frac{2}{3}\right)^2 \cdot S_{31}$

$2\left(\frac{1}{2}\right)\left(\frac{5}{6}\right)^2\left(\frac{2}{3}\right) \cdot S_{41}$ $\quad \frac{2260}{1377}$

$2\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)\left(\frac{2}{3}\right) \cdot S_{42}$ $\quad \frac{56}{81}$

$2\left(\frac{1}{6}\right)\left(\frac{1}{3}\right)\left(\frac{2}{3}\right) \cdot S_{51}$ $\quad \frac{832}{1020}$

$\left(\frac{1}{3}\right)^2\left(\frac{5}{6}\right)^2 \cdot S_{51}$

$2\left(\frac{1}{6}\right)\left(\frac{5}{3}\right)\left(\frac{2}{3}\right) \cdot S_{52}$ $\quad \frac{16}{6561}$

$\left(\frac{1}{6}\right)^2\left(\frac{2}{3}\right)^2 \cdot S_{52}$

$2\left(\frac{1}{6}\right)^2\left(\frac{1}{3}\right)\left(\frac{2}{3}\right) \cdot S_6$ $\quad \frac{2}{81}$

$2\left(\frac{1}{6}\right)\left(\frac{1}{3}\right)^2\left(\frac{5}{6}\right) \cdot S_6$

$S_{31} = \frac{25}{81} S_{31} + \frac{8779441}{2230740}$

S_{32}

$\left(\frac{5}{6}\right)\left(\frac{2}{3}\right)^2\left(\frac{1}{2}\right) \cdot S_{32}$

$\left(\frac{1}{2}\right)\left(\frac{5}{6}\right)\left(\frac{1}{3}\right)^2 \cdot S_{41}$ $\quad \frac{565}{2754}$

$\left(\frac{1}{3}\right)\left(\frac{5}{6}\right)\left(\frac{2}{3}\right) \cdot S_{42}$ $\quad \frac{56}{81}$

$\left(\frac{1}{6}\right)\left(\frac{2}{3}\right)^2\left(\frac{1}{2}\right) \cdot S_{43}$ $\quad \frac{22}{63}$

$\left(\frac{1}{3}\right)\left(\frac{5}{6}\right)\left(\frac{2}{3}\right) \cdot S_{51}$ $\quad \frac{17}{27}$

$\left(\frac{1}{6}\right)\left(\frac{1}{3}\right)\left(\frac{2}{3}\right) \cdot S_{52}$ $\quad \frac{2}{9}$

$\left(\frac{1}{3}\right)^2\left(\frac{5}{6}\right)\left(\frac{1}{2}\right) \cdot S_{52}$

$\left(\frac{1}{6}\right)\left(\frac{1}{3}\right)^2\left(\frac{1}{2}\right) \cdot S_6$.

$\left(\frac{1}{6}\right)\left(\frac{1}{3}\right)\left(\frac{2}{3}\right) \cdot S_6$ $\quad \frac{5}{27}$

$\left(\frac{1}{3}\right)^2\left(\frac{1}{2}\right)\left(\frac{5}{6}\right) \cdot S_6$

$S_{32} = \frac{5}{27} S_{32} + \frac{21095}{6426}$

$S_{32} = \frac{21095}{5236}$

S_{33}

$\frac{16}{81} S_{33}$

$\frac{32}{81} \cdot S_{42}$ $\quad \frac{1792}{1215}$

$\frac{8}{81} \cdot S_{51}$ $\quad \frac{136}{405}$

$\frac{16}{81} \cdot S_{52}$ $\quad \frac{128}{243}$

$\frac{8}{81} \cdot S_6$ $\quad \frac{16}{81}$

$S_{33} = \frac{16}{81} S_{33} + \frac{859}{243}$

S_2

$\left(\frac{5}{6}\right)^3\left(\frac{2}{3}\right)^2 \cdot S_2$

$\left(\frac{1}{3}\right)\left(\frac{5}{6}\right)^3\left(\frac{2}{3}\right) \cdot S_{31}$ $\quad \frac{219486025}{149905728}$

$\left(\frac{1}{3}\right)\left(\frac{5}{6}\right)^2\left(\frac{2}{3}\right)^2 \cdot S_{32}$ $\quad \frac{527579}{127\,2348}$

$\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^2\left(\frac{2}{3}\right)^2 \cdot S_{33}$ $\quad \frac{4275}{18954}$

$\left(\frac{1}{3}\right)^2\left(\frac{5}{6}\right)^2\left(\frac{2}{3}\right) \cdot S_{41}$ $\quad \frac{2825}{5508}$

$\left(\frac{1}{3}\right)^2\left(\frac{5}{6}\right)^3 \cdot S_{41}$

$\left(\frac{1}{6}\right)^2\left(\frac{5}{3}\right)\left(\frac{2}{3}\right)^2 \cdot S_{42}$ $\quad \frac{112}{243}$

$\left(\frac{1}{3}\right)^2\left(\frac{5}{6}\right)^2\left(\frac{4}{3}\right) \cdot S_{42}$

$\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)\left(\frac{2}{3}\right)^2 \cdot S_{43}$ $\quad \frac{55}{1701}$

$\left(\frac{1}{6}\right)^2\left(\frac{1}{3}\right)\left(\frac{5}{6}\right)\left(\frac{4}{3}\right) \cdot S_{51}$ $\quad \frac{17}{243}$

$\left(\frac{1}{9}\right)\left(\frac{1}{3}\right)^2\left(\frac{5}{6}\right) \cdot S_{51}$

$\left(\frac{1}{6}\right)^3\left(\frac{2}{3}\right)^2 \cdot S_{52}$

$\left(\frac{1}{6}\right)^2\left(\frac{1}{3}\right)\left(\frac{5}{3}\right)\left(\frac{4}{3}\right) \cdot S_{52}$ $\quad \frac{20}{243}$

$\left(\frac{1}{6}\right)\left(\frac{1}{3}\right)^3\left(\frac{5}{6}\right) \cdot S_{52}$

$\left(\frac{1}{6}\right)^3\left(\frac{1}{3}\right)\left(\frac{4}{3}\right) \cdot S_6$ $\quad \frac{17}{972}$

$\left(\frac{1}{6}\right)^2\left(\frac{1}{3}\right)^2\left(\frac{5}{2}\right) \cdot S_6$

$S_2 = \frac{125}{486} S_2 + \frac{91\,816\,716\,544}{21\,436\,519\,104}$

Initial

$\frac{15\,625}{46\,656} \cdot S_1$

$\frac{2125}{7776} \cdot S_2$ $\qquad \frac{180\,427\,279\,371\,875}{123\,871\,394\,344\,704}$

$\frac{625}{7776} \cdot S_{31}$ $\qquad \frac{1\,097\,430\,125}{2\,348\,491\,648}$

$\frac{625}{7776} \cdot S_{32}$ $\qquad \frac{13\,184\,375}{40\,715\,136}$

$\frac{65}{15552} \cdot S_{33}$ $\qquad \frac{107\,375}{303\,264}$

$\frac{125}{7776} \cdot S_{41}$ $\qquad \frac{14\,125}{198\,288}$

$\frac{125}{3888} \cdot S_{42}$ $\qquad \frac{175}{1458}$

$\frac{125}{23\,328} \cdot S_{43}$ $\qquad \frac{1375}{81648}$

$\frac{25}{7776} \cdot S_{51}$ $\qquad \frac{85}{7776}$

$\frac{25}{5184} \cdot S_{52}$ $\qquad \frac{25}{1944}$

$\frac{5}{7776} \cdot S_6$ $\qquad \frac{5}{3888}$

$$S_1 = \frac{15\,625}{46\,656} S_1 + \frac{52\,442\,497\,899\,193}{12\,822\,338\,350\,208}$$