How NOT to shuffle cards

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Your typical games of chance, such as blackjack or poker, will involve a third party, the dealer, who shuffles the 52 cards in a standard deck (excluding the jokers). Whilst this role seems trivial, it's in fact much more mathematically significant than you may expect!

There are 52! different permutations in which a deck may be arranged, a number so unfathomably large that it exceeds the number of atoms in the known universe. Given a fair and competent dealer, each of these unique combinations should be as equally as likely. Now imagine a standard game of Texas Hold Em', with 9 players; fewer cards are in use, only about 23, and 26 including burn cards. This means fewer relevant permutations, although the significance of randomness remains. And that raises the question: "What makes a good shuffle?", or more interestingly; "What is the *worst* possible method for shuffling?"

Key Question 0:

What would you consider a *perfect* shuffle?

One way to look at this would be considering how professionals do it. In casinos, dealers will typically either carry out 7 riffle shuffles, or carry it out as follows:

- **Wash Shuffle:** The deck is split in half, and from each half the cards are laid down and mixed amongst each other, such that each face of the card touches the felt once. This is usually done for 4 cycles
- **Riffle Shuffle:** The deck is split into half, and the cards form each half are interleaved with one another
- **Strip Shuffle:** The deck is split into 3 - 5 large chunks and their order is randomised

This is followed by a cut, where an adjacent player or non - playing card is used to randomly rearrange the deck to further prevent cheating. The wash alone is already an incredibly effective shuffle at randomising their order and makes it nearly impossible to track any given card, let alone multiple. If the cards are labelled numbers from 1 - 52, representing their original position, and suppose they went through a wash shuffle, we get:

16	11	6	3	8	1	4	2	22	26	25	21	9
29	50	31	49	40	42	45	28	51	46	41	43	47
24	17	19	5	10	20	7	23	12	15	14	13	18
52	32	37	38	36	34	27	30	35	39	48	33	44

This order seems random, which it is, which just highlights how effective the casino shuffle is. If the goal of the casino is to randomise, then the culmination of these steps has achieved that goal.

Key Question 1:

Imagine you were to compare each method of shuffling.

How would you go about measuring how random a given method is?

The issue with measuring randomness is that we can't quite stick a number on it. You can't measure it because an event is either random, or it isn't. A helpful way to think about randomness is that if you were to record or plot the data, it wouldn't produce a pattern or trend. Take a coin, or a fair die, which regardless of how many times you flip or roll them, you won't ever see something predictable emerge. Mathematicians such as Kolmogorov,ƞƞ who describes randomness as following three axioms:

1. P(E) 0 , probabilities are non-negative
2. P(Ω) = 1 , total probability of events in the sample space is one
3. P(A∪B) = P(A) + P(B) , P(A∩B) = ∅ , probability of multiple disjoint events can be found by adding.

Now how does this apply to our deck of cards? Since each card drawn changes the probability of the next, it's not quite like the coin, or die, which are independent events. However, you'll notice that the cards drawn for a game still follow the three axioms above, so what randomness should instead mean for us is as follows:

<u>Key Question 2:</u>

Suppose you were guessing the cards at the top of a deck, discarding one each time.

How many would you guess correctly, on average?

Using the standard formula for expected probability, we can begin by taking the probability of the first card, $\frac{1}{52}$, and then the probability for the next card, $\frac{1}{51}$, and so on, all the way to the last card. Assuming you've been keeping track of which cards have been discarded, you will guess the last card correctly 100% of the time. So, if we sum all our probabilities, for a perfectly shuffled deck, you should only be able to guess ~4.6 cards correct from the standard deck.

Since we can't measure randomness, this serves as a more suitable metric for quantifying how random something is. If we consider how many repetitions are needed, such that the average number of correct guesses would be reasonably close to 4.6. Any shuffle that needs few repetitions to do so is a "good" shuffle, and vice versa. If we consider the casino shuffle, it seems odd that 7 riffles are also considered acceptable, so why is this?

<u>Riffle Shuffling</u>

The important thing to note about the riffle shuffle is that when it interleaves the two halves of the deck it doesn't alternate the cards perfectly. As such, the position of each card doesn't follow a given pattern. Given that the cards still alternate with the other half, and may even clump together, you might reasonably expect a much higher number of repeats for the deck to be considered "random". But it only takes 7, which is why casinos do so. Bayer & Diaconis, who explored the riffle shuffle in their work "Chasing the Dovetail Shuffle to its Lair", used a metric which they named "variation distance". What this measures is how close to the uniform distribution every permutation of the deck is.

Picture a deck in which you know the order of the cards. If you've shuffled it once, you can easily guess many of the cards correctly, and certain permutations (e.g. ones which place large groups of cards next to each other) are far more likely. As such the variation distance is incredibly low, whilst after having shuffled multiple times, the likelihood of each permutation approaches being perfectly equal to one another. What the two discovered was that the number of shuffles and variation distance could be modelled by a normal distribution, where for the variation distance to be reasonably low, the critical number of riffles needed can be given by m=log2n , where n is the number of cards.

For n = 52, this critical value is ~8.5, although there is a steep transition in variation distance from 5 to 7 shuffles, dropping from 0.924 to 0.331, giving a fairly random deck of cards. This degree of randomness is considered sufficient for casinos, as the variation distance is low enough, whilst balancing fairness with the number of games played.

Variation distance for m shuffles of 52 cards

m	1	2	3	4	5	6	7	8	9	10
Variation Distance	1.000	1.000	1.000	1.000	0.924	0.614	0.334	0.167	0.085	0.000

<u>Wash Shuffling</u>

The wash shuffle serves as a much better method at randomising cards, as the cards are forced past each other. This ends up breaking up groups of cards that were together previously, as well as significantly changing the position of the top & bottom cards, which may not be as dispersed in other methods of shuffling. The shuffle can be modelled using a truncated Poisson distribution, where λ represents the average distance a card has moved from its original position, capped at 51. This shuffle is modelled so that $\lambda = 3$, which is accurate, as the distance a card has moved will be more right skewed.

If you were to picture an individual card, which gets moved and slices through multiple others, you would expect a similar number of cards to have moved atop it, and below it in the deck. This means that the average distance moved from its original position is lower, although each card is also able to move 51 spaces, as much as it is able to move no spaces in the deck. This results in a fairly random shuffle.

<u>Strip/Overhand Shuffling</u>

This method moves large chunks of cards over and over, either to the bottom or top of the deck. Consider a Jenga tower, where players move bricks from the middle to the top of the tower. Even after 10, 20 or 30 turns, the majority of the tower remains unchanged. The same principle applies here, where groups of cards will "stick" to one another, so players will have a higher chance of guessing correctly, since knowing one card means you now know the order of all the cards that follow.

Going back to the Jenga analogy, where the tower does eventually become unrecognisable, this shuffle is also able to reach a point where the deck can be considered "random". However, this number of shuffles is close to 4000, as Pemantle discovered in 1984, where he reached the formula that the number of shuffles required can vary from n2 to n2log(n), so from 2700 to 4000 shuffles may be needed.

<u>Key Question 3:</u>

Do you think a random shuffle is necessarily a good shuffle?

The goal of these previous methods has been to randomise the order of the deck, such that each permutation is equally likely. This means players are unable to accurately guess a majority of the cards but suppose the $\frac{1}{52!}$ where we return to the original position. Then any observant player now knows the order, and consequently position of every relevant card, which you could argue is a worse situation. So maybe, we could instead consider.

"What is the fewest number of shuffles needed to return to its original position?"

For this we can turn to another group of people proficient with cards: magicians. Magicians tend to use one sort of shuffle, that being the perfect faro shuffle. Not too dissimilar from the riffle, the deck is split in half, and each half interleaves each other. The key difference here is that the cards from each half alternate perfectly. Whilst this seems as though it should still produce a perfectly random deck, it follows a given pattern. As such, after you've repeated the shuffle 8 times, you realise that each card has returned to its original position. This is used by magicians for this explicit reason. You place a card randomly, and it looks like they've shuffled it incredibly thoroughly, but after the 8th, nothing has changed, so they can simply go to your card's location.

It is important to note that this only works with out-shuffling, where the top card of the deck and bottom card remain in their respective location, so that when the other cards interleave, they can return to this position in 8 shuffles. When the top card and bottom change their position(to second and fifty-first each time), it takes 52 shuffles to return using perfect Faro "in-shuffles".

Other readers may come to the realisation that there is a simpler way to return the original, following the above conditions, although far easier to track. Suppose you divided the deck into 3 stacks, and shuffled them as follows:

1	1	3	1
2	3	1	2
3	2	2	3

This does not technically undo any shuffles, and returns the deck to its original position in fewer shuffles, hence being a worse shuffle.

Conclusion

If we are concerned with randomness, or players tracking cards, an overhand shuffle especially in large chunks is by far the worst way to shuffle, and should be avoided since it removes the element of chance and luck from card games. More certainly, a poor strip shuffle or Faro shuffle should be avoided further, as they guarantee a return to the starting position, without creating randomness, and hence is the easiest way to get kicked out from the poker game.