How to assume probability in gacha decision
For TRM Essay 2026

1 Introduction

Let's say you play a mobile game. In this game, there's a gacha(loot box) system to get SSSSSR(just an example) item. You have to decide whether to pull it or not. There's certain probability to get SSSSSR item for each chance. You can try multiple times, but chances are limited because most of us are not rich enough to get virtually unlimited chances. So it's important to check the probability of getting item. If it's not high enough, you will give up and skip for next opportunity.

Case 0

There's 2% chance to win(get item) for each chance and you have 50 chances.
What's the probability of winning at least once?

One of the simple ways to guess the result is to see expected value. If situation above is given, then you can easily calculate the expectation by multiplying 2% and 50. Since expectation is 1, you may assume that you can probably get the item. However, it isn't true. Most of us already know that it is not guaranteed to get the item, but only few of us would know that there's more than 35% of failure. That's one of ***expectation fallacy*** that make people think more positively then reality.

$$X_0:\ number\ of\ winnings, \qquad l_k:Losing\ for\ k\ th\ try, \qquad L_k = P(l_k) = 1 - \frac{2}{100}$$

$$\therefore P(X_0 \geq 1) = 1 - P(X_0 = 0) = 1 - L_1 L_2 \cdots L_{49} L_{50} = 1 - \left(1 - \frac{2}{100}\right)^{50} \approx 0.636$$

Since the case is running Bernoulli trials, X_0 is a binomial random variable denoted by $X_0 \sim B(50, \frac{1}{50})$. So it isn't that hard to write answer(the red one) if you know about probability. But it doesn't mean that you can find out the exact result(the green one) right away, because power number is generally too large to calculate manually. That's the reason why expectation fallacy occurs. Perhaps you may get help from calculator or AI to find actual probability.

But what if there's simple formula to get answer without any assistance? For example, ***normal approximation to the binomial*** makes it easy to find probability through standard normal table(Z-table). Similarly, there could be alternative ways to find out results which is really close to actual answer. Then you can check your situation easily.

****normal approximation to the binomial***

Let X be a binomial random variable denoted by $X \sim B(n, p)$.
If n is sufficiently large enough that $np \geq 5$ and $n(1-p) \geq 5$,
then X can be approximated by the normal distribution $N(np, npq)$.

As you see above, expectation itself doesn't really helpful to guess the probability of winning at least once. But expectation is still good indicator because it's literally true that you can get that value on average. So our goal is to find better formula using expectation for speculating probability. From now on, we will try some examples and see patterns through it.

2 Find formula

2.1 Start from one

Since we decided to use expectation for making formula, let expectation as x. If we can make any function $A(x)$ that is really close to actual value, it's done. But since we don't know nothing now, let's start from when $x = 1$(same as case 0).

Case 1

There's *1/n* chance to win for each chance and you have *n* chance(s).
What's the probability of failing to win even once?
$(n \in \mathbb{N})$

Like example from introduction, you can directly find probability using *n*. But the answer itself doesn't give you anything yet. Substitution is the best option for this case. Try some numbers from 1. Since $n = 1$ is too obvious, you can try when $n = 2, 3, 4, 5$

$$X_1 : number\ of\ winnings, \qquad l_k : Losing\ for\ k\ th\ try, \qquad L_k = P(l_k) = \frac{n-1}{n}$$

$$\therefore P(X_1 = 0) = L_1 L_2 \cdots L_{n-1} L_n = \left(\frac{n-1}{n}\right)^n$$

n	2	3	4	5
$P(X_1 = 0) = \left(\frac{n-1}{n}\right)^n$	1/4	8/27	81/256	1024/3125

As you put some numbers, you would notice that the probability get bigger as *n* gets larger. Let $f(n) = \left(\frac{n-1}{n}\right)^n$ $(n \in \mathbb{N},\ n \geq 2)$. Note that $f(n)$ gets bigger as n gets larger, but you don't know $f(n)$ is increasing for all *n* yet. To prove that, you have to show $f(n)$ is increasing function. But it is hard because it is not differentiable. For doing that, you can extend the range of *n* into real numbers. Let $g(x) = \left(\frac{x-1}{x}\right)^x$ $(x \in \mathbb{R},\ x \geq 2)$. Since $g(x)$ is differentiable, if you prove that $g'(x) > 0$ for all $x > 2$ then you can say $f(n)$ is increasing for all $n \geq 2$.

$$g(x) = \left(\frac{x-1}{x}\right)^x$$

The form of $g(x)$ can't let you do $g'(x)$ instantly. For differentiation, take ln on both sides.

$$\ln\{g(x)\} = \ln\left\{\left(\frac{x-1}{x}\right)^x\right\}$$
$$\ln\{g(x)\} = x(\ln(x-1) - \ln x)$$

Since $g(x) \neq 0$ for all $x \geq 2$, now you can differentiate both sides.

$$\frac{g'(x)}{g(x)} = (\ln(x-1) - \ln x) + x\left(\frac{1}{x-1} - \frac{1}{x}\right)$$
$$\frac{g'(x)}{g(x)} = \ln\left(\frac{x-1}{x}\right) + \frac{1}{x-1}$$
$$\therefore g'(x) = g(x) \cdot \left\{\ln\left(\frac{x-1}{x}\right) + \frac{1}{x-1}\right\} = \left(\frac{x-1}{x}\right)^x \left\{\ln\left(\frac{x-1}{x}\right) + \frac{1}{x-1}\right\}$$

Now we have to show $g'(x) > 0$. Since it's obvious that $\left(\frac{x-1}{x}\right)^x > 0$, we need to prove $\ln\left(\frac{x-1}{x}\right) + \frac{1}{x-1} > 0$. We know $\ln\left(\frac{x-1}{x}\right) < 0 < \frac{1}{x-1}$ but we can't solve $\frac{1}{x-1} > \left|\ln\left(\frac{x-1}{x}\right)\right|$ directly. Instead we have to take steps to prove that. Let $h(x) = \ln\left(\frac{x-1}{x}\right) + \frac{1}{x-1}$ $(x \in \mathbb{R},\ x > 2)$.

i) $h(x)$ is differentiable for all $x > 2$ and $\lim_{x\to 2+} h(x) = \ln\frac{1}{2} + 1 > 0$

ii) $h'(x) = \frac{1}{x(x-1)} - \frac{1}{(x-1)^2} < 0$ and $\lim_{x\to\infty} h(x) = 0$

By i), we show $h(x)$ is continuous and it's initial start is positive. And by ii), we find $h(x)$ is decreasing function and it's limit to infinity is 0. Therefore, we can say $h(x) > 0$ for all $x > 2$. It means $g'(x) > 0$ for all $x > 2$.

$\therefore g(x)$ is increasing function

$\therefore f(n)$ is increasing function

(Actually, we can say both functions are ***strictly increasing function***)

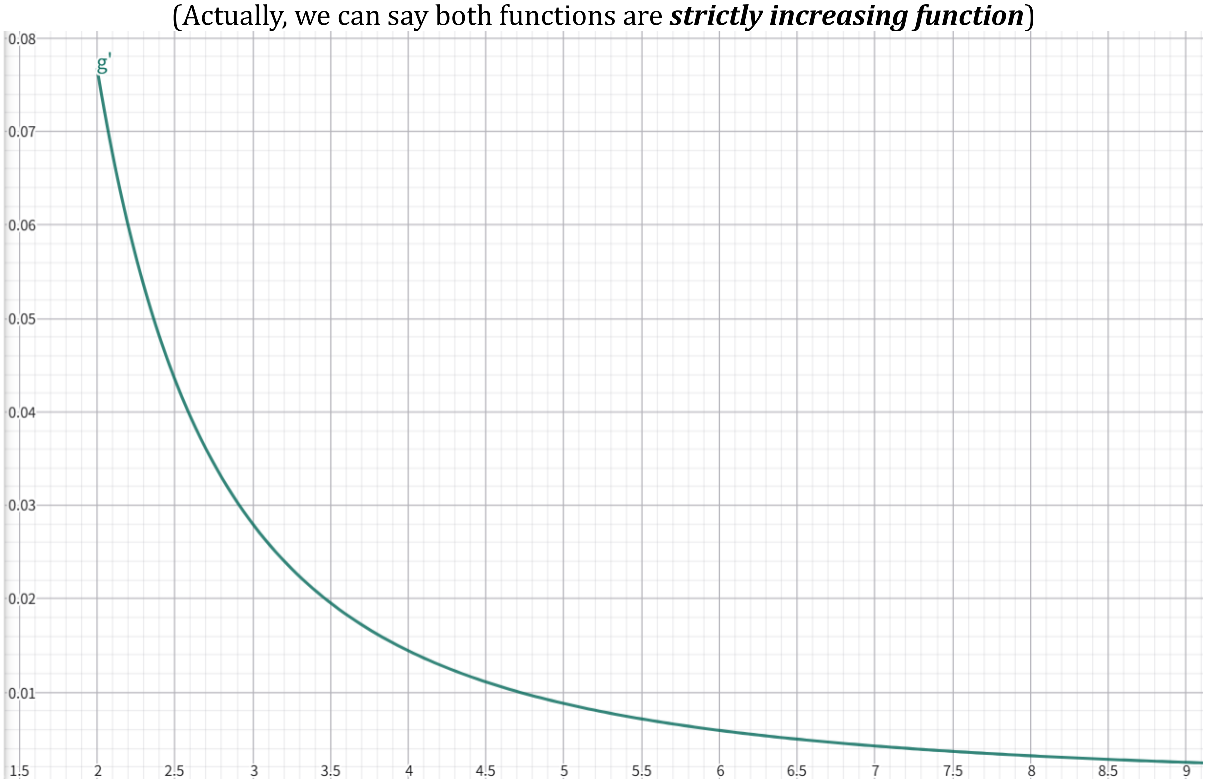

Figure 1 – graph of $y = g'(x)$

Now there's another question. If $f(n)$ is increasing, does it mean we will almost fail when n gets very large? To answer this question, you should find $\lim_{n\to\infty} f(n)$. Fortunately, there's similar well-known form called ***Euler's number***, which is defined by $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$. You can easily get the answer using Euler's number.

$$\therefore \lim_{n\to\infty} f(n) = \lim_{n\to\infty}\left(\frac{n-1}{n}\right)^n = \lim_{n\to\infty}\left\{\frac{1}{\left(\frac{n}{n-1}\right)}\right\}^n = \lim_{n\to\infty}\frac{1^{n-1}}{\left(1+\frac{1}{n-1}\right)^{n-1}}\cdot\left(\frac{n-1}{n}\right) = \frac{1}{e}\times 1 = \frac{1}{e}$$

Existing the limit of the function means it converges. So $f(n)$ is getting closer to *1/e* as *n* gets larger. It implies if *n* is large enough to ignore the difference between real value and *1/e*, we can actually say the probability just *1/e*. What if we try when expectation is 2, 3, or more?

2.2 Generalize

Case 2

There's *1/n* chance to win for each chance and you have *tn* chance(s).
What's the probability of failing to win even once?
$(t, n \in \mathbb{N}, n \neq 1)$

Actually, there's no big difference between case 1 and 2. You can still find $P(X_2 = 0)$ using Euler's number, and $P(X_2 = 0)$ is still increasing and convergent. So I will just show the results.

$$X_2\text{: } number\ of\ winnings, \qquad l_k : Losing\ for\ k\ th\ try, \qquad L_k = P(l_k) = \frac{n-1}{n}$$

$$\therefore P(X_2 = 0) = L_1 L_2 \cdots L_{tn-1} L_{tn} = \left(\frac{n-1}{n}\right)^{tn}$$

$$\therefore \lim_{n\to\infty}\left(\frac{n-1}{n}\right)^{tn} = \lim_{n\to\infty}\left\{\frac{1}{\left(\frac{n}{n-1}\right)}\right\}^{tn} = \lim_{n\to\infty}\left\{\frac{1^{n-1}}{\left(1+\frac{1}{n-1}\right)^{n-1}}\right\}^t\cdot\left(\frac{n-1}{n}\right)^t = \frac{1}{e^t}\times 1^t = \frac{1}{e^t}$$

Then is it the end? Not yet since we don't know proper condition of *n* to use the estimate value. We only said *n* has to be large, but how much? It will be handled soon. Before that, let's write our new theorem based on what we have done. We already know that number of winnings$(\coloneqq X)$ is a binomial random variable denoted by $X \sim B(tn, \frac{1}{n})$. So for mathematical refinement, it would be better to write 'Bernoulli trials' instead of 'gacha situation' in our theorem.

Theorem (in progress)

Assume that there are Bernoulli trials with tn trials and $p = \frac{1}{n}$.
Then $(number\ of\ successes) \coloneqq X$ is a binomial random variable denoted by $X \sim B(tn, \frac{1}{n})$.
If n is sufficiently large enough, we can say $P(X = 0) = e^{-t}$.

2.3 Error rate

Now we only left one step to complete formula. Like explanation of normal approximation to the binomial from introduction, setting appropriate range of n would make our theorem more perfect. We have to find any point that error rate is lower than specific constant.

$$Error\ rate(\%) = \frac{|expected\ value - actual\ value|}{actual\ value} \times 100 = \frac{expected\ value}{actual\ value} \times 100 - 100$$

Since expected value always larger than actual value in our case, error rate can be expressed like above. Next, time to choose target rate that have to be lower than that. I'll set the rate about 1%. You can get that rate when you assume with 50 chances with probability 1/50.

$$1(\%) \geq \frac{\lim_{n\to\infty} P(X_2)}{P(X_2)} \times 100 - 100$$

$$\frac{101}{100} \geq \frac{e^{-t}}{\left(1-\frac{1}{n}\right)^{nt}}$$

$$\ln 101 - \ln 100 \geq -t\left\{1 + n\ln\left(1-\frac{1}{n}\right)\right\}$$

(You can take ln on both sides since we don't consider when n=1)

We found out that one answer of the equation is $n \geq 50$ when $t = 1$. Then how n changes as t get bigger? Suppose that t is doubled, then $\left\{1 + n\ln\left(1-\frac{1}{n}\right)\right\}$ would be halved. But you can't do more with that expression. So this time you have to get help from ***Taylor series***.

Taylor series

If function $f(x)$ is infinitely differentiable at $x = a$,

We can approximate $f(x)$ in different form around $x = a$.

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

It is called the Taylor series of $f(x)$

If we apply with $f(x) = \ln(1-x)$ and $a = 0$, $f(x)$ would be approximated like this.

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = -x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots$$

We can set $x = \frac{1}{n}$ since n is large enough to get close to $x = 0$(in this case, we have $n \geq 50$).

$$\therefore \ln\left(1-\frac{1}{n}\right) = -\frac{1}{n} - \frac{1}{2n^2} - \frac{1}{3n^2} - \cdots$$

So go back to $\left\{1 + n\ln\left(1-\frac{1}{n}\right)\right\}$ and substitute Tylor series for $\ln\left(1-\frac{1}{n}\right)$

$$1 + n\ln\left(1-\frac{1}{n}\right) = 1 + \left(-1 - \frac{1}{2n} - \frac{1}{3n^2} - \frac{1}{4n^3} - \cdots\right) = -\frac{1}{2n} - \frac{1}{3n^2} - \frac{1}{4n^3} - \cdots$$

Remember that we are exploring this topic for gacha situation, where it is common to have more than 100 chances. It means the latter part of the formula $-\frac{1}{3n^2} - \frac{1}{4n^3} - \cdots$ gives little impact on overall result. Therefore only $-\frac{1}{2n}$ is main part, which means n is doubled as t is doubled.

$$-t\left\{1 + n\ln\left(1-\frac{1}{n}\right)\right\} \approx -t \cdot -\frac{1}{2n} = \frac{t}{2n}$$

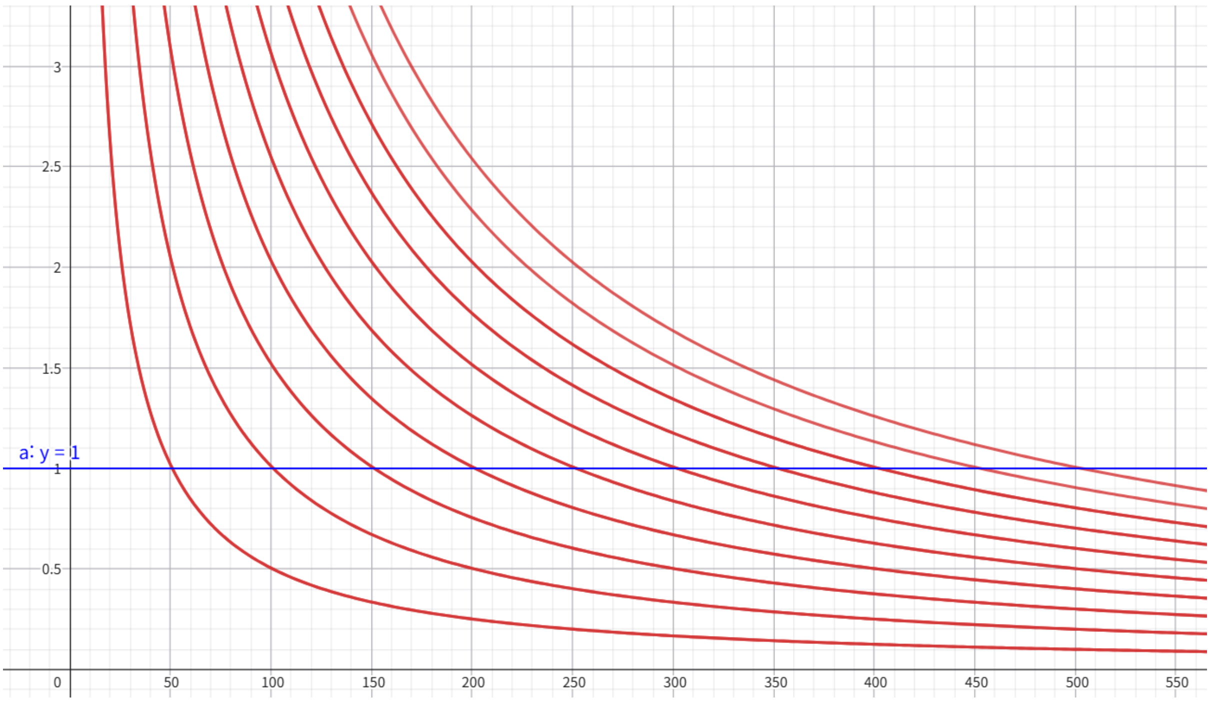

Figure 2 – Error rate(%) when t=1,2,...,10

In conclusion, we can say that if $n \geq 50t$ the error rate would be lower than about 1%.

3 Conclusion

Loot Box Theorem

Assume that there are Bernoulli trials with tn trials and $p = \frac{1}{n}$.

Then $(number\ of\ successes) \coloneqq X$ is a binomial random variable denoted by $X \sim B(tn, \frac{1}{n})$.

If n is sufficiently large enough that $n \geq 50t$, we can say $P(X = 0) = e^{-t}$.

t	1	2	3	4	5
$P(X = 0)$	≈36.8%	≈13.5%	≈5.0%	≈1.8%	≈0.7%

As a result, we finally made it. Although we only find $P(X = 0)$, you can apply to many situations. It doesn't have to be gacha situation. Many of us face probability in reality too. From now you can simply express the possibility of failure using *t*. You don't have to calculate or fall into expectation fallacy. I hope you make good use of it.

4 Extra

Some of you may ask why I don't find $P(X = k)$. In fact, I already found it.

$$P(X = k) = \binom{tn}{k}\left(\frac{1}{n}\right)^k\left(1 - \frac{1}{n}\right)^{tn-k} = \frac{(n-1)^{tn-k}}{n^{tn}} \times \frac{(tn)!}{k!\,(tn-k)!}$$

$$\begin{aligned}
\lim_{n\to\infty} P(X = k) &= \lim_{n\to\infty}\left\{\frac{(n-1)^{tn-k}}{n^{tn}} \times \frac{(tn)!}{k!\,(tn-k)!}\right\} \\
&= \lim_{n\to\infty}\left\{\frac{1}{k!} \times \left(\frac{n-1}{n}\right)^{tn} \times \frac{1}{(n-1)^k} \times \frac{(tn)!}{(tn-k)!}\right\} \\
&= \lim_{n\to\infty}\left\{\frac{1}{k!} \times \left(\frac{n-1}{n}\right)^{tn} \times \frac{tn}{n-1} \times \frac{tn-1}{n-1} \times \frac{tn-2}{n-1} \times \dots \times \frac{tn-k+1}{n-1}\right\} \\
&= \frac{1}{k!} \times \frac{1}{e^t} \times t^k
\end{aligned}$$

But it is really hard to find range of n to lower the error rate. If you interested in this, you can search for it yourself.