How to Calculate Roots without a Calculator

1. Introduction

With modern technology, calculators have been an essential tool for students. Calculators bring mathematicians speed and convenience. However, they have reduced the need for some foundational skills. For example, with graphing calculators, the ability to draw an accurate graph is no longer emphasised. This raises an interesting question: if calculators were taken away, how would we calculate roots?

2. Finding the Square Root

2.1. Trial-and-Error

Method

e.g. Calculating $\sqrt{76}$ to 2 decimal places

1. List the square numbers:

    - 1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144…

2. Identify where the number lies in this list

    - 76 is between 64 and 81, closer to 81

3. Test a range possible values

    - $\sqrt{64}$ is 8 and $\sqrt{81}$ is 9, test values between 8.5 and 9

4. Increase accuracy

    - $8.7^2 = 75.69$ and $8.8^2 = 77.44$
    - As 76 is closer to 75.69, the possible range of $\sqrt{76}$ narrows to 8.70 to 8.75

5. Find the answer to the appropriate decimal places

    - $8.71^2 = 75.8641$ and $8.72^2 = 76.0384$
    - As 76 is closer to 76.0384, $\sqrt{76} = 8.72$ (2 d.p.)

Advantages and limitations

Advantages: It is the easiest way to calculate a root. It is also the most widely accessible because it does not require advanced mathematical knowledge.

Limitations: It becomes inefficient for larger numbers.

2.2. Advanced Trial and Error (Simplifying & Trial and Error)

Method

e.g. Calculating $\sqrt{1900}$ to 2 decimal places

1. Simplify $\sqrt{1900}$ by short division:

   - 1900 can be written as $2^2 \times 5^2 \times 19$

   - $\therefore \sqrt{1900} = \sqrt{2^2(5^2)(19)} = 10\sqrt{19}$

2. Apply the Trial-and-Error Process

   - 19 lies between 16 and 25, so $\sqrt{19}$ is between 4 and 5

   - Test values from 4.1 to 4.5

     - $4.3^2 = 18.49$ and $4.4^2 = 19.36$

     - As 19 is closer to 19.36, test values from 4.35 to 4.39

   - $4.35^2 = 18.9225$ and $4.36^2 = 19.0096$

     - 19 is closer to 19.0096, try to square 4.355 to 4.359

   - $4.358^2 = 18.992164$ and $4.359^2 = 19.000881$

   - As 19 is closer to 19.000881, $\sqrt{19} = 4.359$ (3 d.p.)

3. Multiplication

   - $10 \times 4.359 = 43.59$ (2 d.p.)

Advantages and limitations

Advantages: Through simplifying, it creates a smaller root, which reduces the number of trials needed and increases the speed. It is also not a difficult way, so it is accessible.

Limitations: It does not work when numbers cannot be simplified neatly.

2.3. Digit-by-Digit Method

Method

e.g. Calculating √ 1900 to 2 decimal places

1. Put the digits into pairs (if the number has an odd number of digits, leave the left-most digit unpaired, and if the number has decimals, pair up decimal digits instead of with the whole-number digits)

   - Put 1900 into pairs, 1 and 9 and 0 and 0

2. List the square numbers:

   - 1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144…

3. Choose the largest square number that is smaller than or equal to the first pair of numbers

   - Pick 16, as it is the largest square number that is smaller than 19

4. Do the calculation using the "long division" method, the root of the square number picked would be the divisor

   - ```
        4
     4 √ 19 00
         16

          3 00
     ```

5. Multiply the partial quotient by 2, times the product x by 10 and plus y (the next digit(s) of the partial quotient)

   - $4\ (2) = 8$
   - $x = 8; 8\ (10) = 80$
   - $80 + y = 80 + y$

6. Set up an equation:
     ```

$$(10x + y)(y) \leq \textit{remainder of the partial long division}$$

- $(80 + y)\ (y) \leq 300$

7. Find y
   - When $y = 3$, $83(3) = 249$; When $y\ = 4$, $84\ (4) = 336$
   - Therefore, $y = 3$

8. Repeat the Long division, with $10x + y$ being the divisor

   -
   ```
           4 3
      4 √ 19 00
          16
   83      3 00
           2 49
             51
   ```

9. Repeat steps 5 to 8 until suitable decimal places

   -
   ```
           4 3  . 5   8  …
      4 √ 19 00. 00 00
          16
   83      3 00
           2 49
   865       51  00
             43  25
   8708        7  75 00
               6  96 64
                   78 36
   ```

   - √ 19 00 = 43.58 (2 d.p.)

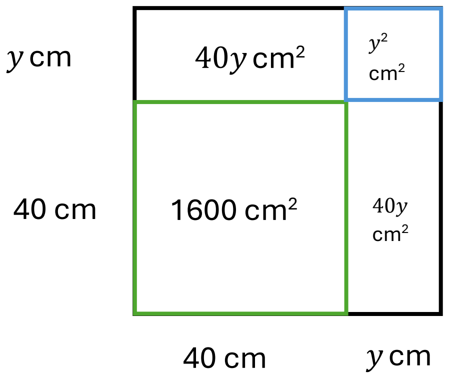
Explanation

To explain it graphically:

Let the area of the square be 1900 cm^2

Let the side of the green square be 40 cm, the area of green square be 1600 cm^2

Let the side of the black square be $(40 + y)$ cm



From the graph,

$$1600 + 2(40y) + y^2 = 1900$$

$$1600 + y((2(40)) + y) = 1900$$

$$y\left((2(40)) + y\right) = 300$$

$y((2(40)) + y) = (80 + y)(y) = 300$ matches with Step 5 and Step 6

Multiply the partial quotient by 2, times the product x by 10 and plus y (the next digit(s) of the partial quotient)

Set up an equation:

$$(10x + y)(y) \leq \textit{ remainder of the partial long division}$$

- $4(2)(10) + y = 80 + y$
- $(80 + y)(y)$

This geometric model explains why the digit-by-digit method works and how each step is constructed.

Advantages and limitations

Advantages: It has a systematic way of doing the calculations, which makes this method faster. It can also work for large numbers efficiently.

Limitations: It required memorisation of the method, it is not as intuitive as the other methods.

2.4. Newton-Raphson Method

Method

e.g. Calculating √ 1900 to 2 decimal places

1. Simplify √ 1900 by short division:

   - 1900 can be written as 2^2 x 5^2 x 19

   - $\therefore \sqrt{1900} = \sqrt{2^2(5^2)(19)} = 10\sqrt{19}$

2. List the square numbers:

   - 1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144…

3. Choose the largest square number that is smaller than or equal to the first pair of numbers

- Pick 16, as it is the largest square number that is smaller than 19

4. Find the difference between the number to be square rooted and the square number picked

- $19 - 16 = 3$

5. Set up an equation (let *x* be the number to be square rooted, *y* be the root of the square number picked

$$\sqrt{x} \approx y + \frac{\mathrm{x} - \mathrm{y}^2}{2\mathrm{y}}$$

- $\sqrt{19} \approx 4 + \frac{3}{2(4)}$
- $\sqrt{19} \approx 4.375$

6. Repeat Step 4 with the square of the approximation calculated be the square number picked for a more accurate answer

- $19 - 4.375^2 = -0.140625$

7. Repeat Step 5 with $y =$ The approximation calculated

- $\sqrt{19} \approx 4.375 + \frac{-0.140625}{2(4.375)}$
- $\sqrt{19} \approx 4.358928571$

8. Repeat Steps 6 and 7 to the appropriate accuracy
9. Multiplication

- $\sqrt{19} \approx 4.358928571$
- $10\sqrt{19} \approx 43.59$ (2 d.p.)

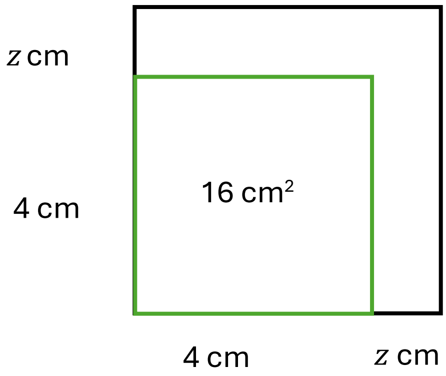
Explanation

To explain it graphically:

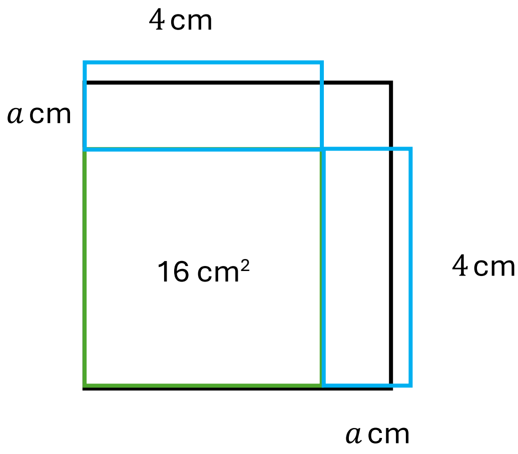
Let the area of the square be 19 cm^2, the side of the black square be $\sqrt{19}$ cm

Let the side of the green square be 4 cm, the area of the green square be 16 cm^2, the maximum area with an integer as the side of the green square

Let the side of the black square be $(\sqrt{19} - 4 = z)$ cm



Knowing that the area of the black L-shape = 19 – 16 = 3 cm, rearranging it into two blue rectangles with a length of 4 cm and an unknown width of a cm



From the graph,

$$2(4a) + 16 \approx 19$$

$$8a + 16 \approx 19$$

$$8a \approx 3$$

$$a \approx \frac{3}{8}$$

The side of the black square ≈ The side of the green square + the width of the blue rectangle

$$4 + a \approx \sqrt{19}$$

$$4 + \frac{3}{8} \approx \sqrt{19}$$

This corresponds exactly to the Newton–Raphson update formula in Step 5.

Steps 6 and 7 represent refining the approximation by adjusting the width of the rectangles, making the estimate more accurate each time.

Advantages and limitations

Advantages: It is extremely fast, making it the most efficient method. It also provides the answer to a lot of decimal places in a short period of time.

Limitations: It required memorisation of the method, and it is not as intuitive as the other methods.

3. Finding the Nth Root

3.1. Trial and Error

Method

- Same as 2.1 –

3.2. Using Taylor Expansion

Method

e.g. Calculating $\sqrt[4]{96}$ to 2 decimal places

1. List to the nth power numbers

   - 1, 16, 81, 256, 625…

2. Choose the largest to the nth power number that is smaller than the number to be nth rooted, and let it be x

   - $x = 81$, as it is the largest to the 4th power number that is smaller than 96

3. Use the equation:

   $N =$ The number to be nth rooted
   $x =$ The root of the chosen perfect power
   $n =$ The index of the root

$$\sqrt[n]{N} \approx \frac{(n-1)x^n + N}{n(x)^{n-1}}$$

- $n = \sqrt[4]{81} = 3$
- $\sqrt[4]{96} \approx \frac{(4-1)3^4 + 96}{4(3)^{4-1}}$
- $\sqrt[4]{96} \approx 3.13$ (2 d.p.)

Explanation

To explain this method, an explanation of the equation above should be mentioned. In an n^th root $\sqrt[n]{N}$ can be rewritten to $\sqrt[n]{x^n + y}$, where x^n =the nearest perfect nth power below *N*, $y = N - x^n$, a small positive integer

$$\sqrt[n]{x^n + y}$$

$$= x^n \sqrt[n]{1 + \frac{y}{x^n}}$$

As $\frac{y}{x^n}$ is very small, the first-order Taylor expansion can be used,

$$f(\delta) = \sqrt[n]{1 + \delta}$$

The Taylor expansion around $\delta = 0$ is:

$$\sqrt[n]{1 + \delta} \approx 1 + \frac{\delta}{n}.$$

Substitute $\delta = \frac{y}{x^n}$:

$$\sqrt[n]{x^n + y} \approx x\left(1 + \frac{y}{nx^n}\right).$$

Put $y = N - x^n$ into the equation,

$$x(1 + \frac{y}{nx^n})$$

$$= x(1 + \frac{N - x^n}{nx^n})$$

$$= x\left(\frac{nx^n}{nx^n} + \frac{N - x^n}{nx^n}\right)$$

$$= x\left(\frac{nx^n - x^n + N}{nx^n}\right)$$

$$= x(\frac{x^n(n-1) + N}{nx^n})$$

$$= \frac{x^n(n-1) + N}{nx^{n-1}}$$

General Equation:

$$\frac{x^n(n-1) + N}{nx^{n-1}}$$

Advantages and limitations

Advantages: It is fast, because it requires only substitution. It works for any n^{th} roots.

Limitations: It required knowledge of Taylor expansion. It is not accurate when *y* is large.

3.3. Using Binomial Expansion (to find the approximation)

Method

e.g. Calculating $\sqrt[4]{96}$ to 2 decimal places

1. List to the n^{th} power numbers

- 1, 16, 81, 256, 625…

2. Trial and Error to 1 decimal place to find an underestimate, *x*

- Observe where the number to 4^{th} root is in this list
    - 96 is between 81 and 256, closer to 81

- Try a range of numbers
    - $\sqrt[4]{81}$ is 3 and $\sqrt[4]{256}$ is 4, try squaring numbers starting from 3.1 to 3.5
    - After trying, $3.1^4 = 92.3521$ and $3.2^4 = 104.8576$

- Find x
    - As $92.3521 < 96$, $x = 3.1$

3. Use the equation:

   $N =$ The number to be nth rooted
   $y =$ The difference between the estimated number and the actual number
   $x =$ The estimated root
   $n =$ The index of the root

$$y \approx \frac{N - x^n}{n(x)^{n-1}}$$

   - $y \approx \frac{96 - 3.1^4}{4(3.1)^{4-1}}$
   - $y \approx 0.0306124 \ldots$

4. Add the difference and the estimated number together to get the answer

   - $\sqrt[4]{96} \approx x + y$
   - $\sqrt[4]{96} \approx 3.1 + 0.0306124$
   - $\sqrt[4]{96} \approx 3.13$ (2 d.p.)

Explanation

To explain this method, an explanation of the equation above should be mentioned.

$x =$ The first estimation of $\sqrt[4]{96}$ (which can be found using the trial and error method)

$y =$ The difference between the estimated number and the actual number

$$\sqrt[4]{96} \approx x + y$$

$\sqrt[4]{96}$ = 3.1 + a small difference between the estimated number and the actual number

$$\left(\sqrt[4]{96}\right)^4 \approx (x + y)^4$$

Use Binomial Expansion to expand $(x + y)^4$

$$(x + y)^4$$

$$= x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$

$6x^2y^2 + 4xy^3 + y^4$ is a small number that the calculation can ignore,

$$(x+y)^4 \approx x^4 + 4x^3y$$
$$96 \approx x^4 + 4x^3y$$
$$96 \approx x^3(x+4y)$$
$$\frac{96}{x^3} - x \approx 4y$$
$$y \approx \frac{96}{4x^3} - \frac{x}{4}$$
$$y \approx \frac{96-x^4}{4x^3}$$

General Formula:

$$y \approx \frac{N - x^n}{n(x)^{n-1}}$$

Advantages and limitations

Advantages: It is fast, because it requires only substitution. It works for any n^{th} roots.

Limitations: It requires a good initial estimation with the trial-and-error method, which takes time. In the Binomial Expansion, some terms are ignored, which affects the accuracy.

4. Conclusion

In this essay, several methods for calculating roots without a calculator have been explored, including Trial-and-Error, Digit-by-Digit, and the Newton–Raphson Method for square roots, as well as Trial-and-Error, using Taylor Expansion, and Binomial Expansion for nth roots.

A summary of the methods is shown in the tables below:

Square Roots:

	Speed	Accuracy	When to use
Trial-and-Error	Slow	Medium	Estimation for small numbers
Advanced Trial-and-Error	Medium	Medium	When the numbers have large square factors
Digit-by-Digit Method	Medium	High	When high accuracy is needed; For large numbers
Newton–Raphson Method	Fast	High	When fast, accurate results are needed; For large numbers

Nth Roots:

	Speed	Accuracy	When to use
Taylor Expansion Method	Fast	Medium-High	When the number is near a perfect nth power
Binomial Expansion Method	Medium-fast	High	When you have a good first estimate

Ultimately, these methods show that even without calculators, mathematics can still provide effective methods for calculating roots. Only by studying different calculations without using calculators can the true beauty of mathematics be appreciated.

References

https://youtu.be/6evC4klO_lI?si=meULwo4UlvOjO539

https://youtu.be/MXveVqBxFow?si=4lfUJssKvCcHBU7b

https://youtu.be/6ArrxekYR2U?si=fSQ5LZQL-BMMN5vQ

https://youtu.be/20UKj6N-e0s?si=f-Au1vAT2SD8OqRs