How Maps Lie to Us: The Coastline Paradox

Nizar Ismael

April 2026

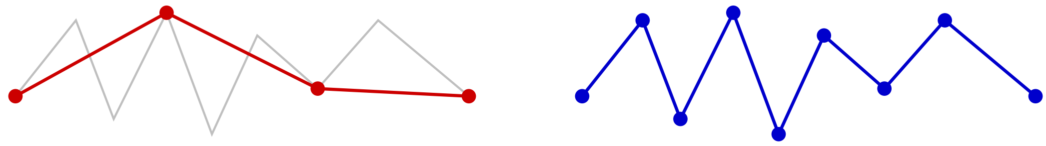

Figure 1: The same coastline measured with a long ruler (left) and a short ruler (right). The finer the ruler, the greater the total measured length.

Introduction

I was born in Dubai in 2010, by which point the Palm Jumeirah had already risen fully from the Gulf; an artificial coastline engineered into existence about three years before I was. Off its coast, in 2001, engineers had started construction on it: An ambitious project, it was designed deliberately in the shape of a palm tree, this was done not only to pay homage to the culture of the UAE, but by making the shape more intricate and branching, they had multiplied the coastline nearly sevenfold.

In doing this, they had stumbled onto a deeply rooted problem without even realising it.

Ask someone how long a coastline is and they may say, "Get a map, and measure it." However, the answer you get entirely depends on how you measure, use a long ruler and you miss the small bays and inlets; use a shorter one and they appear. The shorter your ruler, the longer the coastline appears, with no obvious limiting point.

This is the coastline paradox and it forces us to ask a question that sounds almost philosophical: "What does length mean?"

1 Richardson's Ruler

The first person to take this problem seriously was Lewis Fry Richardson, a British scientist who stumbled upon it almost by accident. He was investigating whether the length of shared borders between countries correlated with the likelihood of war when he spotted something he couldn't ignore: Spain and Portugal had reported completely different lengths for their shared border. Not approximately different, completely different. This suggested something deeper than just a measurement error.

Richardson started measuring coastlines and borders systematically, using different ruler lengths each time. Every time he reduced the ruler size, the length increased, and not randomly, it followed a pattern. If you call the ruler length ε and the measured length $\mathrm{L}(\varepsilon)$. Richardson found that:

$$\mathrm{L}(\varepsilon) \sim \varepsilon^{1-D}$$

Where D is a constant that represents the geometric complexity of a coastline, a quantity that we will define shortly. Taking the logarithm of both sides:

$$\log \mathrm{L}(\varepsilon) \sim (1-D)\log \varepsilon$$

So if you plot $\log L$ against $\log \varepsilon$ you get a straight line with slope $1-D$. For a smooth curve like a circle, $D = 1$ and the slope is zero (the length converges no matter how small your ruler gets). But for the coast of Britain, Richardson found $D \approx 1.25$. The slope is negative, meaning that the measured length keeps growing as ε shrinks without any limiting value.

That number (1.25), is not a whole number. And that's the problem, what does it mean for something to have a dimension that is not an integer?

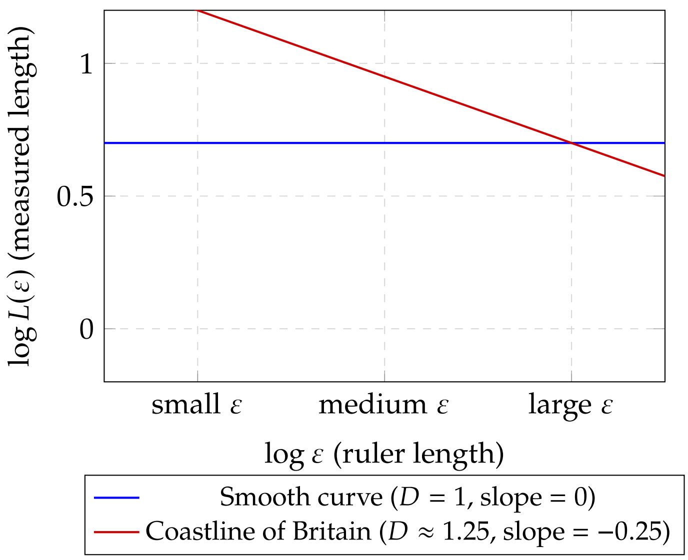

Figure 2: Log-log plot of measured length $L(\varepsilon)$ against ruler length ε. A smooth curve has slope zero; a fractal coastline has a strictly negative slope with gradient $1-D$.

2 From Integers to Fractals: The Hausdorff Dimension

So far we have a number; $D \approx 1.25$ for the British coastline, but we have not yet said what it actually is. It clearly isn't a standard dimension in the way we usually mean. A line is one-dimensional, a square is two-dimensional, nothing in ordinary geometry lies between the two. So where does the 1.25 come from, and what is it measuring?

To answer this, we need to think differently about what dimension means. The classical way to define an object is to ask: how does it scale? If you double the side length of a line segment, its "size" doubles (it scales by a factor of 2^1). Double the side length of a square and its area scales by 2^2 (similarly, a cube scales by 2^3). In general, for a d-dimensional object scaled by a factor r, the "size" scales as r^d, leaving us with familiar integer results.

Now consider the Koch snowflake; a simple example of a fractal, a shape that is self-similar at every scale (zoom in on any section and it looks like the whole). You construct it by forming an equilateral triangle, then repeatedly replacing the middle third of each line segment with two sides of another equilateral triangle. At each step, 1 segment becomes 4, each of length $\frac{1}{3}$ the original. So scaling by $\frac{1}{3}$ gives you 4 copies, applying the same logic as before:

$$(\frac{1}{3})^d = \frac{1}{4} \implies d = \frac{\log 4}{\log 3} \approx 1.26$$

(Note that this d is the Hausdorff dimension for the Koch snowflake specifically; Britain's coastline has its own value $D \approx 1.25$ that Richardson had calculated empirically rather than analytically, although their values are remarkably similar).

Not 1, not 2, but somewhere in between. This is the Hausdorff dimension; a precise, well-defined quantity that encodes the geometric complexity of a shape by measuring how it fills space as you zoom in.

The coastline isn't quite a line and it isn't quite a surface, it is somewhere in between. The Hausdorff dimension enables us to determine exactly how in between.

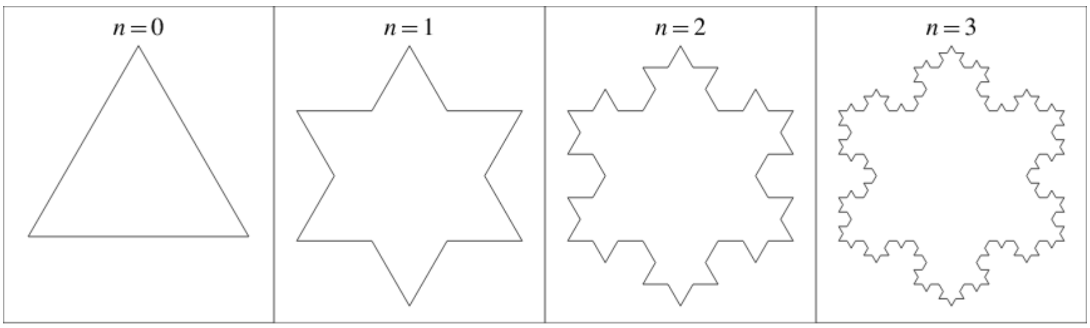

Figure 3: Construction of the Koch snowflake, showing increasing complexity at each iteration. Source: Wolfram MathWorld.

3 Measure Theory: A New Yardstick

The Hausdorff Dimension tells us that a coastline has a fractional dimension; but it does not yet tell us how to assign it a meaningful "size". Length clearly fails, as we've seen; area fails too as a coastline has zero area by any standard definition. So what is the right way to measure something that sits between dimensions?

This is the question that measure theory was built to answer. Measure theory is essentially the branch of mathematics that aims to assign a notion of size to any arbitrary set of points. Ordinary examples such as length, area and volume do not work for fractional dimensions, so we need something more flexible.

Intuitively, instead of trying to measure a shape directly, we can try to "cover" it with small pieces and we can see what happens when these pieces get smaller and smaller. Imagine placing discs of radius r over a coastline until the entire thing is covered, if you start to shrink r, the number of discs needed to cover the coastline increases, and how fast this number increases depends on the shape we are measuring. A straight line needs roughly $\frac{1}{r}$ discs; a square needs roughly $\frac{1}{r^2}$ discs and in general, a fractal needs $\frac{1}{r^D}$ discs for some $D : 1 < D < 2$, where D encodes how the shape fills space, this is exactly the Hausdorff dimension appearing naturally.

To make this mathematically precise, we introduce some new notation: a ball $B_{r_i}(x_i)$ centered at x_i with radius r_i is the set of all points of distance r_i away from x_i (in two dimensions this represents a disc centered at x_i with radius r_i). So formally, the s-dimensional Hausdorff measure on a set E is defined as:

$$\mathcal{H}^s(E) = \lim_{\delta \to 0} \inf \{ \sum_i r_i^s : E \subseteq \bigcup_i B_{r_i}(x_i), \quad r_i < \delta \}$$

Where we write $E \subseteq \bigcup_i B_{r_i}(x_i)$, we mean that the union of these discs completely cover our set E. We take the infimum of the set to ensure the most efficient covering possible, and we force $\delta \to 0$ to ensure that these discs shrink sufficiently (we are asking what happens to the limit of infinitely fine coverings).

4 Conclusion

We started with a seemingly trivial question: how long is a coastline? However, the answer pulls us through some of the deepest ideas in modern mathematics; Richardson's ruler experiments, the geometry of fractals, and the rigor of measure theory. The Palm Jumeirah, built by engineers who intuitively understood that branching multiplies length, sits at the centre of a problem that took mathematicians decades to formalise.

What the coastline paradox teaches us is not that measurement is impossible, but that the definition of "measurement" can change depending on what you are measuring. Instinctively we reach for a ruler, however a ruler assumes whatever we are measuring is one-dimensional, when it isn’t the ruler lies. And every map contains this lie exactly: it has chosen a scale, fixed a ruler length, and presented a result to us as if it were a fact.

I am of Syrian heritage; my family is from a country with its own Mediterranean coastline, one I knew mostly from maps rather than memory. I grew up trusting those maps without ever questioning what they had left out. The coastline paradox has taught me that every map has to make a compromise between reality and simplicity, the smaller your ruler, the more reality you recover, but you never do recover all of it.

This is what I find most remarkable about this problem. Mathematics doesn’t just expose the lie, it also gives us the tools to see through it, and it tells us the degree to which we have been misled. The Hausdorff dimension measures the gap between the map and the territory. Measure theory tells us what a meaningful notion of size even looks like for something that lies in between dimensions. "How long is a coastline?" turns out to be the wrong question to ask. What we should start to ask is "What does size even mean here?" And in chasing that, from the engineered fronds of an island I grew up beside to the battered edges of the British coast, we built an entirely new theory of size.